String Theory on AdS_3
as
Discrete Light-Cone Liouville Theory

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Abstract

We investigate (super) string theory on AdS_3 background based on an approach of free field realization. We demonstrate that this string theory can be reformulated as a string theory defined on a linear dilaton background along the transverse direction ("Liouville mode") and compactified onto S^1 along a *light-like* direction.

Under this reformulation we analyze the physical spectrum as that of a free field system, and discuss the consequences when we turn on the Liouville potential. The discrete light-cone momentum in our framework is naturally interpreted as the "winding number" of the long string configuration and is identified with the spectral flow parameter that is introduced in the recent work by Maldacena and Ooguri [11].

Moreover we show that there exist infinite number of the on-shell chiral primary states possessing the different light-cone momenta and the spectral flow consistently acts on the space of chiral primaries. We observe that they are also chiral primaries in the sense of space-time (or the conformal theory of long string) and the spectrum of space-time $U(1)_R$ charge is consistent with the expectation from the AdS_3/CFT_2-duality. We also clarify the correspondence between our framework and the symmetric orbifold theory of multiple long string system [17].

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1 Introduction

Study of string theory on AdS_3 background has been a subject of great importance mainly for the following two reasons: Firstly, it is a non-trivial example of completely solvable string theories on a curved background with the Lorentzian signature [1, 2, 3]. The second reason, which is comparably newer than the first one, is the possibility of understanding the AdS/CFT-duality [4] at a stringy level [5, 6, 7].

Although the string theory on this background (with NS B-field) is believed to be described by a simple conformal field theory, $SL(2; \mathbf{R})$-WZW model[5], there are still subtleties. Especially two different theoretical grounds have been proposed with respect to the set up of the Hilbert space of quantum states;

1. The Hilbert space is defined to be a representation space of current algebra $\widehat{SL}(2; \mathbf{R})$.

2. The Hilbert space is defined to be the Fock space of free fields which should be identified with the string coordinates in some suitable parametrizations of the group manifold $SL(2; \mathbf{R})$.

The first ground is based on the standard prescription of two dimensional conformal field theory (CFT_2). It is well-known that the WZW model for a compact group actually has such a Hilbert space: the physical Hilbert space should be made up of a finite number of the unitary (integrable) representations of current algebra. Since we now have a non-compact group $SL(2; \mathbf{R})$, the situation becomes rather non-trivial. If $k \equiv k' - 2$ (k' means the level of $\widehat{SL}(2; \mathbf{R})$) is equal to a negative rational value, we have a finite number of the "admissible" representations[1] which contain a rich structure with many singular vectors [8]. However, string theory on AdS_3 background corresponds to the cases of *positive k*, in which we cannot have any unitary representation of $\widehat{SL}(2; \mathbf{R})$ (except for the trivial representation). There are infinite number of non-unitary representations some of which have a few singular vectors (generically, no singular vectors). The best we can expect is that the BRST condition successfully eliminates all negative norm states from the physical Hilbert space. Many discussions have been given concerning the no-ghost theorem along this line [1]. With respect

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[1]The admissible representations are not necessarily unitary representations of $\widehat{SL}(2; \mathbf{R})$ with $k = -q/p$. But these define the good-natured conformal blocks in the similar manner as the familiar (p, q) minimal model with $c = 1 - \dfrac{6(p-q)^2}{pq}$ [8].

to the discrete series the no-ghost theorem was proved under the assumption of a truncation of quantum number "j" parametrizing the second Casimir [9].

In this traditional approach to CFT_2 from the representation theory of current algebra, primary states are naively characterized by the two quantum numbers j, m. However, as was carefully discussed by Bars [2], there is a subtle point if we recall that we are working in a σ-model *with a three dimensional non-compact target space.* One should keep it in mind that under the limit of weak curvature; $k \longrightarrow +\infty$, AdS_3 space (the universal covering of AdS_3, strictly speaking) may be replaced by a flat background $\mathbf{R}^{2,1}$. In this sense it may be more natural that we have *three* conserved momenta characterizing the physical states, and the second definition of Hilbert space is likely to be more appropriate, as was claimed in the works [2, 3, 10].

More recently, based on the representation theory of $\widehat{SL}(2; \mathbf{R})$, Maldacena and Ooguri claimed [11] that one should take the Hilbert space enlarged by the spectral flow and proved the extended no-ghost theorem. On physical ground it may be plausible that the third momentum we mentioned above is related to such an enlargement of Hilbert space. One of the main purposes of this paper is to clarify the relation between them, namely, the role of spectral flow in the argument of [11] and that of the third momentum of zero-modes [2, 3, 10] (discrete light-cone momentum in the context of this paper).

Another aim of this work is to manifest further the role of "space-time Virasoro algebra" introduced in [5]. It is inspired by the asymptotic isometries of Brown-Henneaux [12, 13] and is understood to describe the conformal symmetry of the long string sector [14, 15]. The generators of this algebra are most conveniently realized as operators acting on the Fock space of free fields (the Wakimoto's φ, β, γ system in the usual treatment). This is one of the reasons why we take the free field realization rather than the abstract representation theory of current algebra.

This paper is organized as follows;

In section 2 we start with reformulating the bosonic and superstring theories on AdS_3 background by a free field realization. With the help of some field redefinitions we show that the AdS_3 string theory can be described by a string theory on a linear dilaton background (along the transverse direction) and with a *light-like compactification.* We further demonstrate that the space-time conformal algebra given in [5] has a quite simple form analogous to the

DDF operators [16] in this framework of "discrete light-cone Liouville theory".

In section 3 we analyze the physical spectrum in our framework. As that of a free field system we will reproduce the spectrum proposed in [2, 3, 10]: Only the principal series is allowed due to the unitarity. We also comment on the outcome of turning on the Liouville potential term, which should be a marginal perturbation. Such an interaction term breaks the translational invariance along the radial direction (in other words, makes the "screening" of the extra zero-mode momentum from the view points of the $SL(2; \mathbf{R})$ current algebra), and hence the "bound string states" possessing the imaginary momentum along this direction may appear in the physical spectrum. The space-time Virasoro operators play a role as the *spectrum generating algebra*, and we will observe that one must include the representations of $\widehat{SL}(2; \mathbf{R})$ which are broader than those given in [11] in order to make the full space-time Virasoro algebra act successfully on the physical Hilbert space.

In section 4 we further investigate the spectrum for superstring cases that give rise to space-time $N = 2, 4$ SCFTs. We present the complete set of on-shell chiral primaries. We will find that there are infinite number of on-shell chiral primary states with the different light-cone momenta, and the spectral flows act naturally among them. They become the chiral primaries also in the sense of *space-time* and have the space-time $U(1)_R$ charges in agreement with the expectation of AdS_3/CFT_2-duality. As a byproduct we also clarify the relationship with the symmetric orbifold CFT describing the multiple long strings discussed in [17].

We will summarize the main results of our analyses and present some discussions in section 5.

2 Reformulation of AdS_3 String Theory as the Discrete Light-cone Liouville Theory

2.1 Bosonic String on $AdS_3 \times \mathcal{N}$

Through this paper we shall consider the universal covering of the AdS_3 space with the Lorentzian signature so that the time direction is non-compact. We start with the following

world-sheet Lagrangian for the (quantum) bosonic string on AdS_3 with NS B-filed [5]

$$\mathcal{L} = \partial\varphi\bar{\partial}\varphi - \sqrt{\frac{2}{k}}R^{(2)}\varphi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \beta\bar{\beta}\exp\left(-\sqrt{\frac{2}{k}}\varphi\right), \tag{2.1}$$

where $R^{(2)}$ denotes the curvature on the world-sheet. Throughout this paper we shall only focus on the physics at the near boundary region $\varphi \sim +\infty$, in which we can consider the interaction term ("screening charge term") $\beta\bar{\beta}\exp\left(-\sqrt{\frac{2}{k}}\varphi\right)$ as a small perturbation. By dropping this term simply we obtain the free conformal field theory

$$T = -\frac{1}{2}\partial\varphi\partial\varphi - \frac{1}{\sqrt{2k}}\partial^2\varphi + \beta\partial\gamma, \tag{2.2}$$

$$\varphi(z)\varphi(0) \sim -\ln z, \quad \beta(z)\gamma(0) \sim \frac{1}{z}. \tag{2.3}$$

We will later discuss the effect of restoring this interaction term $\beta\bar{\beta}\exp\left(-\sqrt{\frac{2}{k}}\varphi\right)$ on the physical spectrum.

The $SL(2;\mathbf{R})$ symmetry in this free system is described by the Wakimoto representation [18]

$$
\begin{aligned}
j^- &= \beta \\
j^3 &= \beta\gamma + \sqrt{\frac{k}{2}}\partial\varphi \\
j^+ &= \beta\gamma^2 + \sqrt{2k}\gamma\partial\varphi + (k+2)\partial\gamma,
\end{aligned}
\tag{2.4}
$$

which generates the $\widehat{SL}(2;\mathbf{R})$ current algebra of level $k+2$

$$
\begin{cases}
j^3(z)j^3(0) &\sim -\dfrac{(k+2)/2}{z^2} \\
j^3(z)j^\pm(0) &\sim \dfrac{\pm 1}{z}j^\pm(0) \\
j^+(z)j^-(0) &\sim \dfrac{k+2}{z^2} - \dfrac{2}{z}j^3(0).
\end{cases}
\tag{2.5}
$$

By using the standard bosonization of β, γ [19]

$$\beta = i\partial v e^{-u-iv}, \quad \gamma = e^{u+iv}, \quad u(z)u(0) \sim -\ln z, \ v(z)v(0) \sim -\ln z, \tag{2.6}$$

we can rewrite the currents (2.4)

$$
\begin{aligned}
j^- &= i\partial v e^{-u-iv} \\
j^3 &= -\partial u + \sqrt{\frac{k}{2}}\partial\varphi \\
j^+ &= e^{u+iv}(k\partial(u+iv) + \sqrt{2k}\partial\varphi + i\partial v).
\end{aligned}
\tag{2.7}
$$

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Moreover it is convenient to introduce the following new variables;

$$
\begin{aligned}
Y^0 &:= \sqrt{\frac{2}{k+2}}\,iu - \sqrt{\frac{k}{k+2}}\,i\varphi \\
Y^1 &:= -\sqrt{\frac{k+2}{2}}\,v + \frac{k}{\sqrt{2(k+2)}}\,iu + \sqrt{\frac{k}{k+2}}\,i\varphi \\
\rho &:= \sqrt{\frac{k}{2}}(u+iv) + \varphi \ .
\end{aligned}
\tag{2.8}
$$

Since this field redefinition is an $SO(2,1)$-rotation, we simply have

$$
Y^0(z)Y^0(0) \sim \ln z \ , \quad Y^1(z)Y^1(0) \sim -\ln z \ , \quad \rho(z)\rho(0) \sim -\ln z \ ,
\tag{2.9}
$$

and any other combinations have no singular OPEs.

In these variables the $\widehat{SL}(2;\mathbf{R})_{k+2}$ currents are given by

$$
\begin{aligned}
j^3 &= \sqrt{\frac{k+2}{2}}\,i\partial Y^0 \\
j^\pm &= \left(-\sqrt{\frac{k+2}{2}}\,i\partial Y^1 \pm \sqrt{\frac{k}{2}}\,\partial\rho \right) e^{\mp\sqrt{\frac{2}{k+2}}\,i(Y^0+Y^1)} \ ,
\end{aligned}
\tag{2.10}
$$

and also the stress tensor is rewritten as

$$
T = \frac{1}{2}(\partial Y^0)^2 - \frac{1}{2}(\partial Y^1)^2 - \frac{1}{2}(\partial\rho)^2 - \frac{1}{\sqrt{2k}}\partial^2\rho \ ,
\tag{2.11}
$$

which of course has the correct central charge $c = 3 + \dfrac{6}{k}$.

In this way we have found that the bosonic string theory on AdS_3 can be realized by two free bosons Y^0, Y^1 (with no background charge) and a "Liouville mode" ρ with the background charge $Q \equiv \sqrt{\dfrac{2}{k}}$. The essentially same realizations of $\widehat{SL}(2;\mathbf{R})$ were used in several works [20, 3, 21]. It was suggested in [22] that the fields Y^0, Y^1, ρ roughly corresponds to the global coordinates of AdS_3 space, and in the similar sense one might suppose that the Wakimoto fields φ, β, γ correspond to the Poincaré coordinates.

The definitions of Hermitian conjugations of these free fields are standard

$$
(Y^i(z))^\dagger = Y^i(1/z), \quad (\rho(z))^\dagger = \rho(1/z) - Q\ln z.
\tag{2.12}
$$

(Recall that the Liouville field ρ has a background charge.) One can easily verify that the Hermitian conjugations of the current generators take the usual forms $j_n^{3\dagger} = j_{-n}^3$, $j_n^{\pm\dagger} = j_{-n}^\mp$

under (2.12). This is an advantage of this free field realization (2.10) compared with the Wakimoto representation in which the rules of Hermitian conjugation are not simple.

Notice also that the OPE of Y^0 with itself has the wrong sign (2.9) and thus Y^0 should correspond to the time-like coordinate. This fact is consistent with the usual interpretation; $j_0^3 \sim$ energy.

There is a subtle point with respect to the realizations of currents (2.10): $j^{\pm}(z)$ are not necessarily defined as local operators on the whole Fock space of Y^0, Y^1, ρ. To overcome this difficulty it is natural to assume the following light-like compactification. Let us introduce the light-cone coordinates

$$Y^{\pm} = \frac{1}{\sqrt{2}}(Y^0 \pm Y^1) , \tag{2.13}$$

and assume the periodic identification

$$Y^- \sim Y^- + 2\pi R , \quad R = \frac{2}{\sqrt{k+2}} . \tag{2.14}$$

Such a prescription is known as the name "discrete light-cone quantization", and have been applied to the studies of M(atrix) theory with finite N [23].

In this case, the conjugate momentum of Y^- is quantized as

$$\frac{\partial Y^+}{\partial \tau} \equiv P^+ + \bar{P}^+ = \frac{2n}{R} , \quad n \in \mathbf{Z} , \tag{2.15}$$

and the winding mode of Y^- should take

$$\frac{\partial Y^-}{\partial \sigma} \equiv P^- - \bar{P}^- = mR , \quad m \in \mathbf{Z} . \tag{2.16}$$

Since Y^+ remains non-compact, there is no winding mode along this direction

$$\frac{\partial Y^+}{\partial \sigma} \equiv P^+ - \bar{P}^+ = 0 , \tag{2.17}$$

and thus we obtain $P^+ = \bar{P}^+ = \frac{n}{R}$ ($n \in \mathbf{Z}$).

By using these facts, we can concretely write the "tachyon" vertex operators $\mathcal{V}_{j,m,\bar{m},p}(z,\bar{z}) \equiv V_{j,m,p}(z)\bar{V}_{j,\bar{m},p}(\bar{z})$, where the left mover is defined by

$$V_{j,m,p} = e^{\left(\frac{\sqrt{k+2}}{2}p - \frac{2m}{\sqrt{k+2}}\right)iY^+ + \frac{\sqrt{k+2}}{2}piY^- - \sqrt{\frac{2}{k}}j\rho} , \tag{2.18}$$

and the winding condition (2.16) means that $m - \bar{m} \in \mathbf{Z}$. The corresponding Fock vacuum $|j, m, p\rangle$ has the following properties;

$$j_0^3|j, m, p\rangle = (m - \frac{k+2}{2}p)|j, m, p\rangle, \tag{2.19}$$

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$$j_{\mp p}^{\pm}|j, m, p\rangle = (m \pm j)|j, m \pm 1, p\rangle, \tag{2.20}$$

$$j_{\mp p+n}^{\pm}|j, m, p\rangle = 0, \quad (\forall n \geq 1), \tag{2.21}$$

$$L_0|j, m, p\rangle = \left(-\frac{1}{k}j(j-1) + mp - \frac{k+2}{4}p^2\right)|j, m, p\rangle. \tag{2.22}$$

Namely, j, m mean the quantum numbers appearing in the usual $\widehat{SL}(2; \mathbf{R})$ theory and p corresponds to the label of "flowed representation" of [11]. In fact, the spectral flow in the context of [11] is defined as the following transformations

$$\begin{cases} j^3(z) \longrightarrow j^3(z) + \dfrac{k+2}{2}\dfrac{p}{z} \\ j^{\pm}(z) \longrightarrow z^{\mp p} j^{\pm}(z). \end{cases} \tag{2.23}$$

In the system of Y^0, Y^1, ρ, this is simply the momentum shift

$$Y^0 \longrightarrow Y^0 - p\sqrt{\frac{k+2}{2}}i \ln z, \tag{2.24}$$

and Y^1, ρ remain unchanged.

The global $SL(2; \mathbf{R})$ algebra $\{j_0^3, j_0^{\pm}\}$ is manifestly BRST invariant. We can immediately extend this algebra to the "space-time Virasoro algebra"

$$\begin{cases} \mathcal{L}_0 = -j_0^3 \equiv -\oint \sqrt{\dfrac{k+2}{2}}i\partial Y^0 \\ \mathcal{L}_n = \oint \left(\sqrt{\dfrac{k+2}{2}}i\partial Y^1 - n\sqrt{\dfrac{k}{2}}\partial\rho\right) e^{-\frac{2n}{\sqrt{k+2}}iY^+} \quad (n \neq 0), \end{cases} \tag{2.25}$$

which actually generates the Virasoro algebra on the Fock space over the vacuum $|j, m, p\rangle$

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} , \tag{2.26}$$

where $c = 6(k+2)p$.

We here make a few comments: Firstly, the Virasoro operators $\mathcal{L}_n$ are well-defined as local operators on the whole Fock space compatible with the light-like compactification (2.14). Secondly, it is natural to regard $\mathcal{L}_n$ ($n \neq 0$) as analogs of the DDF operators [16] along the ρ-direction. In fact, $\mathcal{L}_n$ ($n \neq 0$) is no other than the unique solution for the BRST condition among the operators having the form $\sim \oint (A\partial\rho + B\partial Y^+ + C\partial Y^-)e^{-\frac{2n}{\sqrt{k+2}}iY^+}$, $A \neq 0$ (up to BRST exact terms and some overall constant, of course). In the next section we will make use of such DDF like operators in order to construct the complete set of the physical states. As the last comment, we should point out that $\mathcal{L}_n$ are BRST equivalent to the space-time

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Virasoro operators constructed in [5]. It is straightforward to confirm that the quantum number p precisely coincides with the "winding of γ"; $\oint \gamma^{-1} d\gamma = p$.

2.2 Superstring on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$

Let us try to extend our previous results to the superstring cases. We start with the general superstring vacua $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ studied in [24], which are compatible with the world-sheet $N = 2$ SUSY. The most familiar example $AdS_3 \times S^3 \times M^4$ ($M^4 = T^4$ or $K3$) is nothing but a special example of these backgrounds, and we can readily apply the results in this subsection to that case too.

First of all, to fix the notations we summarize the world-sheet properties of this superstring model;

- AdS_3 sector (j^A, ψ^A)

  To extend to the superstring case, we introduce free fermions in the adjoint representation

  $$\psi^3(z)\psi^3(0) \sim -\frac{1}{z} \quad , \quad \psi^+(z)\psi^-(0) \sim \frac{2}{z} ,$$
  $$\psi^\pm = \psi^1 \pm i\psi^2 . \tag{2.27}$$

  The total $\widehat{SL}(2;\mathbf{R})$ currents are given by

  $$J^A = j^A + j^A_F = j^A - \frac{i}{2}\epsilon^A{}_{BC}\psi^B\psi^C , \quad A, B, C = 1, 2, 3 , \tag{2.28}$$

  where the fermionic currents j^A_F have the level -2. The fermionic currents j^A_F can be written by free fermions as

  $$j^\pm_F = \pm\psi^\pm\psi^3 , \quad j^3_F = \frac{1}{2}\psi^+\psi^- . \tag{2.29}$$

  This sector has an $N = 1$ superconformal symmetry given by

  $$G_{SL(2,\mathbf{R})} = \sqrt{\frac{2}{k}}(\frac{1}{2}\psi^+ j^- + \frac{1}{2}\psi^- j^+ - \psi^3 j^3 - \frac{1}{2}\psi^+\psi^-\psi^3) , \tag{2.30}$$

  and the central charge is

  $$c = \frac{3(k+2)}{k} + \frac{3}{2} . \tag{2.31}$$

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- S^1 sector (Y, χ)

  We have a scalar field Y parametrizing S^1

  $$Y(z)Y(0) \sim -\ln z, \tag{2.32}$$

  and its fermionic partner χ

  $$\chi(z)\chi(0) \sim \frac{1}{z} \ . \tag{2.33}$$

  This sector has the simplest $N = 1$ superconformal symmetry

  $$G_{S^1} = \chi i \partial Y \ , \tag{2.34}$$

  with the central charge

  $$c = \frac{3}{2} \ . \tag{2.35}$$

- $\mathcal{N}/U(1)$ sector

  We require that this sector has an $N = 2$ superconformal symmetry described by the currents

  $$T_{\mathcal{N}/U(1)} \ , \quad G^{\pm}_{\mathcal{N}/U(1)} \ , \quad J_{\mathcal{N}/U(1)} \ . \tag{2.36}$$

  The relation between $N = 2$ and $N = 1$ superconformal current is

  $$G_{\mathcal{N}/U(1)} = G^{+}_{\mathcal{N}/U(1)} + G^{-}_{\mathcal{N}/U(1)} \ . \tag{2.37}$$

  Because of the criticality condition, the central charge of this sector should be equal to

  $$c = 9 - \frac{6}{k} \ , \tag{2.38}$$

  and the $U(1)_R$ current satisfies

  $$J_{\mathcal{N}/U(1)}(z)J_{\mathcal{N}/U(1)}(0) \sim \frac{3 - \frac{2}{k}}{z^2} \ . \tag{2.39}$$

We can realize the $N = 2$ superconformal symmetry on the world-sheet in this system. We choose the $U(1)_R$ current as

$$J_R = J_{R1} + J_{R2} + J_{\mathcal{N}/U(1)} \ , \tag{2.40}$$

where

$$J_{R1} = \frac{1}{2}\psi^{+}\psi^{-} + \frac{2}{k}J^3 \tag{2.41}$$

$$J_{R2} = \chi\psi^3 \ . \tag{2.42}$$

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According to the charge of this current the $N = 1$ superconformal current splits into two terms

$$
\begin{aligned}
G &= G_{SL(2,\mathbf{R})} + G_{S^1} + G^+_{\mathcal{N}/U(1)} + G^-_{\mathcal{N}/U(1)} \\
&\equiv G^+ + G^- ,
\end{aligned} \tag{2.43}
$$

where

$$
G^\pm = G^\pm_1 + G^\pm_2 + G^\pm_{\mathcal{N}/U(1)} \tag{2.44}
$$

$$
G^\pm_1 = \frac{1}{\sqrt{2k}}\psi^\pm j^\mp \tag{2.45}
$$

$$
G^\pm_2 = \frac{1}{\sqrt{2}}(\chi \mp \psi^3)\left(\frac{1}{\sqrt{2}}i\partial Y \pm \frac{1}{\sqrt{k}}J^3\right) . \tag{2.46}
$$

The energy-momentum tensor is also decomposed as follows;

$$
T = T_1 + T_2 + T_{\mathcal{N}/U(1)} \tag{2.47}
$$

$$
T_1 = \frac{1}{k}(j^A j_A + J^3 J^3) - \frac{1}{4}(\psi^+ \partial\psi^- - \partial\psi^+ \psi^-) \tag{2.48}
$$

$$
T_2 = -\frac{1}{k}J^3 J^3 - \frac{1}{2}(\partial Y)^2 - \frac{1}{2}\chi\partial\chi + \frac{1}{2}\psi^3\partial\psi^3. \tag{2.49}
$$

It may be worthwhile to mention that the superconformal generators $\{T_i, G^\pm_i, J_{Ri}\}$ (anti-)commute among the different sectors. Furthermore $\{T_1, G^\pm_1, J_{R1}\}$ has the same expression as that of the Kazama-Suzuki coset model for $SL(2;\mathbf{R})/U(1)$ [25].

The BRST charge Q_{BRST} is defined in the standard manner

$$
Q_{BRST} = \oint \left[c\left(T - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi - \eta\partial\xi + \partial cb\right) + \eta e^\phi G - b\eta\partial\eta e^{2\phi}\right], \tag{2.50}
$$

where ϕ, η, ξ are the familiar bosonized superghosts [19].

Now let us try to reformulate this superstring model as the discrete light-cone Liouville theory as in the case of bosonic string. Our goal is the $N = 2$ Liouville theory [26] with the light-like compactification; $\mathbf{R}_+ \times S^1_- \times \mathbf{R}_\rho \times S^1 \times \mathcal{N}/U(1)$. To this aim we need to perform further field redefinitions.

As a preliminary we bosonize the fermions $\psi^\pm$

$$
\psi^\pm = \sqrt{2}e^{\pm iH_1} , \tag{2.51}
$$

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where $H_1(z)H_1(0) \sim -\ln z$, and the radius of compact boson H_1 should be 1. Let Y_0, Y_1 be as given in (2.8), and define

$$
\begin{aligned}
X^0 &:= \sqrt{\frac{k+2}{k}}Y^0 + \sqrt{\frac{2}{k}}H_1 \\
X^1 &:= -\frac{2}{\sqrt{k(k+2)}}Y^0 + \sqrt{\frac{k}{k+2}}Y^1 - \sqrt{\frac{2}{k}}H_1 \\
H_1' &:= \sqrt{\frac{2}{k+2}}(Y^0 + Y^1) + H_1 \ .
\end{aligned}
\tag{2.52}
$$

Since this is again an $SO(2,1)$ rotation, we have the OPEs

$$
X^0(z)X^0(0) \sim \ln z , \quad X^1(z)X^1(0) \sim -\ln z , \quad H_1'(z)H_1'(0) \sim -\ln z ,
\tag{2.53}
$$

and all the non-diagonal OPEs vanish. We also rewrite

$$
X^2 := Y , \quad \Psi^2 := \chi , \quad \Psi^0 := \psi^3 , \quad \Psi^{\pm}(\equiv -\frac{1}{\sqrt{2}}(\Psi^1 \pm i\Psi^\rho)) := e^{\pm iH_1'} \ .
\tag{2.54}
$$

After all, we have changed the system of

$$
\{\varphi, \beta, \gamma, Y, \psi^{\pm}, \psi^3, \chi\}
\tag{2.55}
$$

into the system of new free fields

$$
\{\rho, X^0, X^1, X^2, \Psi^{\pm}, \Psi^0, \Psi^2\} \ .
\tag{2.56}
$$

In these new variables the energy-momentum tensors (2.48), (2.49) are rewritten as

$$
T_1 = -\frac{1}{2}(\partial X^1)^2 - \frac{1}{2}(\partial \rho)^2 - \frac{1}{\sqrt{2k}}\partial^2 \rho - \frac{1}{2}(\Psi^+ \partial \Psi^- - \partial \Psi^+ \Psi^-)
\tag{2.57}
$$

$$
T_2 = \frac{1}{2}(\partial X^0)^2 - \frac{1}{2}(\partial X^2)^2 + \frac{1}{2}\Psi^0 \partial \Psi^0 - \frac{1}{2}\Psi^2 \partial \Psi^2.
\tag{2.58}
$$

The $U(1)_R$ currents (2.41), (2.42) become

$$
J_{R1} = \Psi^+ \Psi^- - Qi\partial X^1
\tag{2.59}
$$

$$
J_{R2} = -\Psi^0 \Psi^2 \ .
\tag{2.60}
$$

Q is the background charge of Liouville mode ρ and in this case $Q = \sqrt{\frac{2}{k}}$.

The $N = 2$ superconformal currents (2.45), (2.46) now take the following forms

$$G_1^\pm = -\frac{1}{\sqrt{2}}\Psi^\pm(i\partial X^1 \pm \partial\rho) \mp \frac{Q}{\sqrt{2}}\partial\Psi^\pm \tag{2.61}$$

$$G_2^\pm = -\frac{1}{\sqrt{2}}(\Psi^0 \mp \Psi^2) \times \frac{1}{\sqrt{2}}i\partial(X^0 \pm X^2) \ . \tag{2.62}$$

It is also convenient to rewrite the total super current

$$G = -\Psi^0 i\partial X^0 + \Psi^1 i\partial X^1 + \Psi^2 i\partial X^2 + \Psi^\rho i\partial\rho + Qi\partial\Psi^\rho. \tag{2.63}$$

Notice that $\{T_1, G_1^\pm, J_{R1}\}$ (2.48), (2.45), (2.41) have been now transformed into the expressions of superconformal algebra in the $N = 2$ Liouville theory [26] as we mentioned before. The essential part of this field redefinition is the identification between $SL(2;\mathbf{R})/U(1)$ Kazama-Suzuki model and the $N = 2$ Liouville theory (see the appendix B of [27], and also refer [28]) and it was claimed in [29] that these two theories are related by a T-duality.

As in the bosonic case, we introduce the light-cone coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1), \tag{2.64}$$

and assume the compactifications

$$X^- \sim X^- + \frac{4\pi}{\sqrt{k}}, \quad H_1' \sim H_1' + 2\pi. \tag{2.65}$$

These are indeed consistent with the previous compactifications $Y^- \sim Y^- + \dfrac{4\pi}{\sqrt{k+2}}$, $H_1 \sim H_1 + 2\pi$, because we can obtain from (2.52)

$$\begin{cases} X^- = \dfrac{2}{\sqrt{k(k+2)}}Y^+ + \sqrt{\dfrac{k+2}{k}}Y^- + \dfrac{2}{\sqrt{k}}H_1 \\[2mm] H_1' = \dfrac{2}{\sqrt{k+2}}Y^+ + H_1 \ . \end{cases} \tag{2.66}$$

We likewise introduce the tachyon vertices compatible with this light-like compactification

$$V_{j,m,p} = e^{\left(\frac{\sqrt{k}}{2}p - \frac{2m}{\sqrt{k}}\right)iX^+ + \frac{\sqrt{k}}{2}piX^- - \sqrt{\frac{2}{k}}j\rho}, \tag{2.67}$$

and the corresponding Fock vacua satisfying

$$J_0^3|j,m,p\rangle = (m - \frac{k}{2}p)|j,m,p\rangle, \tag{2.68}$$

$$J_{\mp p}^\pm|j,m,p\rangle = (m \pm j)|j,m \pm 1,p\rangle, \tag{2.69}$$

$$J_{\mp p+n}^\pm|j,m,p\rangle = 0, \quad (\forall n \geq 1), \tag{2.70}$$

$$L_0|j,m,p\rangle = \left(-\frac{1}{k}j(j-1) + mp - \frac{k}{4}p^2\right)|j,m,p\rangle. \tag{2.71}$$

The total $SL(2; \mathbf{R})$ currents are also rewritten in the new coordinates;

$$J^3 = \sqrt{\frac{k}{2}} i\partial X^0 \tag{2.72}$$

$$J^\pm = \left(-\sqrt{\frac{k}{2}} i\partial X^1 \pm \sqrt{\frac{k}{2}} \partial\rho - \Psi^+\Psi^- \pm \sqrt{2}\Psi^0\Psi^\pm \right) e^{\mp\frac{2}{\sqrt{k}}iX^+}. \tag{2.73}$$

To close this section, we present the space-time superconformal algebra in our new variables, which again has the forms reminiscent of the DDF operators.

First, the space-time Virasoro algebra is given (in the (-1)-picture) by

$$\begin{cases} \mathcal{L}_0 = -\sqrt{\frac{k}{2}} \oint e^{-\phi}\Psi^0 \\ \mathcal{L}_n = \sqrt{\frac{k}{2}} \oint e^{-\phi} e^{-n\frac{2}{\sqrt{k}}iX^+} (\Psi^1 + ni\Psi^\rho) \quad (n \neq 0). \end{cases} \tag{2.74}$$

We again mention that $\mathcal{L}_n$ $(n \neq 0)$ is the unique solution of the BRST constraint among the operators of the form $\sim \oint e^{-\phi} e^{-n\frac{2}{\sqrt{k}}iX^+} (A\Psi^\rho + B\Psi^0 + C\Psi^1)$, $A \neq 0$.

The space-time $U(1)_R$ current is given by

$$\mathcal{J}_n = \sqrt{2k} \oint e^{-\phi} e^{-n\frac{2}{\sqrt{k}}iX^+} \Psi^2 . \tag{2.75}$$

To construct the space-time super currents we must introduce the spin fields. According to [24], we bosonize the "deformed $U(1)_R$ current" on the world-sheet;

$$\begin{aligned} J'_R &:= J_R - Qi\partial(X^0 + X^2) \\ &\equiv i\partial H'_1 - i\partial H_2 - i\sqrt{3}\partial H_3 - Qi\partial(X^0 + X^1) , \end{aligned} \tag{2.76}$$

where we set

$$\begin{aligned} i\partial H'_1 &= \Psi^+\Psi^- \quad \text{(as defined before)} \\ i\partial H_2 &= \Psi^0\Psi^2 \\ -i\sqrt{3}\partial H_3 &= J_{\mathcal{N}/U(1)} - Qi\partial X^2 . \end{aligned} \tag{2.77}$$

(The combined current $J_{\mathcal{N}/U(1)} - Qi\partial X^2$ actually has the Schwinger term $\sim \frac{3}{z^2}$.) The practical reason why we do so is as follows: J_R has non-trivial OPEs with the vertex operators such as $e^{-n\frac{2}{\sqrt{k}}iX^+}$, and thus the DDF like operators including the spin fields (see (2.81)) made up of J_R do not nicely behave under the BRST transformation. In contrast, we can rather simply solve the BRST condition for the vertices associated to the current J'_R, since it has no singular OPE with $e^{-n\frac{2}{\sqrt{k}}iX^+}$.

Now the spin fields should take the form (up to cocycle factors)

$$e^{\frac{i}{2}(\epsilon_1 H_1' + \epsilon_2 H_2 + \sqrt{3}\epsilon_3 H_3)} e^{\epsilon_4 \frac{i}{2}Q(X^0 + X^1)} \quad (\epsilon_i = \pm 1) . \tag{2.78}$$

The GSO projection leaves only a half of them satisfying

$$\prod_{i=1}^{3} \epsilon_i = -1 \quad , \tag{2.79}$$

and we use the notation

$$S^{\epsilon_3 \epsilon_1} = e^{\frac{i}{2}(\epsilon_1 H_1' + \epsilon_2 H_2 + \sqrt{3}\epsilon_3 H_3)} \tag{2.80}$$

to express the spin fields allowed by the GSO condition. We can explicitly verify that the fermion vertices of the type $\oint e^{-\frac{\phi}{2}} e^{\frac{i}{2}\epsilon_4 Q(X^0 + X^1)} S^{\epsilon_3 \epsilon_1}$ are BRST invariant, if and only if $\epsilon_4 = -\epsilon_1$, namely the vertices of the type $\oint e^{-\frac{\phi}{2}} e^{-\frac{2}{\sqrt{k}}\frac{\epsilon_1}{2}iX^+} S^{\epsilon_3 \epsilon_1}$. They define the space-time $N = 4$ SUSY (the global part of the space-time $N = 2$ superconformal symmetry in NS sector). We can work out more general fermion vertices of the type

$$\sim \oint e^{-\frac{\phi}{2}} e^{-r\frac{2}{\sqrt{k}}iX^+} \sum_{\epsilon_3, \epsilon_1} A_{\epsilon_3, \epsilon_1} S^{\epsilon_3 \epsilon_1}, \quad (r \in \frac{1}{2} + \mathbf{Z}) . \tag{2.81}$$

The BRST condition uniquely (up to BRST exact terms and an overall normalization) determines the coefficients $A_{\epsilon_3, \epsilon_1}$ and we finally obtain the physical vertices

$$\mathcal{G}_r^{\pm} = k^{1/4} \oint e^{-\frac{\phi}{2}} e^{-r\frac{2}{\sqrt{k}}iX^+} \left[(r + \frac{1}{2})S^{\pm +} - (r - \frac{1}{2})S^{\pm -} \right] , \quad (r \in \frac{1}{2} + \mathbf{Z}) . \tag{2.82}$$

They generate the $N = 2$ superconformal algebra (in NS sector) together with $\mathcal{L}_n$, $\mathcal{J}_n$, and the central charge is equal to $6kp$ on the Fock space over the vacuum $|*, *, p\rangle$. In fact, it is a straightforward calculation to check that these space-time superconformal generators (2.74), (2.75), (2.82) coincide with the ones constructed in [24] up to BRST exact terms.

3 Analyses on Spectra of Physical States

In this section we shall investigate the spectrum of physical states in our reformulation of AdS_3 string theory. The unitarity of physical Hilbert space is an important problem. We first analyze the physical spectrum as that of *a free field theory*, namely without taking the Liouville potential term into account, and later discuss the effect of turning on this term.

3.1 Spectrum as Free Field Theory

First, we analyze the spectrum of bosonic string on $AdS_3 \times \mathcal{N}$. Let us recall that the Fock vacuum is defined from the tachyon vertex

$$|j, m, p\rangle = \lim_{z \to 0} V_{j,m,p}(z)|0\rangle \ , \tag{3.1}$$

and we denote the corresponding Fock space as $\mathcal{F}_{j,m,p}$ from now on. We also define "out state" as

$$\langle j, m, p| = \lim_{z \to \infty} \langle 0|V_{j,m,p}(z)z^{2h_{j,m,p}} \ , \tag{3.2}$$

where

$$h_{j,m,p} = -\frac{1}{k}j(j-1) + mp - \frac{k+2}{4}p^2 \ . \tag{3.3}$$

Using the momentum conservation and taking account of the existence of background charge along the ρ-direction, we obtain

$$\langle 1-j, -m, -p|j, m, p\rangle \neq 0 \ , \tag{3.4}$$

and the other combinations vanish. Notice that we must use the following Hermitian conjugation

$$\langle 1-j, -m, -p| = (|j, m, p\rangle)^\dagger \tag{3.5}$$

to discuss the unitarity.

We shall here neglect the Liouville potential term (that is, the "screening charge term" $\int \beta\bar{\beta}e^{-\sqrt{\frac{2}{k}}\varphi} \sim \int \partial Y^1 \bar{\partial} Y^1 e^{-\sqrt{\frac{2}{k}}\rho}$ in the σ-model action (2.1)). This means that the Hilbert space of physical states should be defined as the BRST cohomology on the Fock spaces of the free fields Y^0, Y^1, ρ properly tensored by the Hilbert space of $\mathcal{N}$ sector. Deciding the physical spectrum is a rather simple problem as in the usual string theory on the flat Minkowski space (at least as long as we only take the primary states in the $\mathcal{N}$ sector in constructing the physical states). However, there is one non-trivial point: the existence of background charge in the ρ-direction. As is well-known, the BRST constraint eliminates two longitudinal degrees of freedom in the case of Minkowski background: one is eliminated by the BRST condition itself and another becomes BRST exact. From a physical point of view this aspect is closely related to the fact that the first excited states (the graviton states) become mass-less, and thus one of the polarization vectors must be light-like. On the other hand, in our case of linear

dilaton theory we have a mass gap originating from the background charge of ρ and so the first excited states become *massive*. This implies that one of the two longitudinal modes does *not* decouple from the physical Hilbert space, since all the polarization vectors are space-like.

To make this point clearer, let us consider a simple example. We are here given one transverse oscillator $i\partial\rho = \sum_n \dfrac{\alpha_n^\rho}{z^{n+1}}$ and two longitudinal oscillators $i\partial Y^\pm = \pm\sum_n \dfrac{\alpha_n^\pm}{z^{n+1}}$. The simplest candidate of the first excited states is

$$\alpha_{-1}^\rho|p^+, p^-, p^\rho\rangle \otimes c_1|0\rangle_{\text{gh}}, \tag{3.6}$$

where the on-shell condition is given by

$$p^+ p^- + \frac{1}{2}(p^\rho)^2 + \frac{1}{4k} = 0 \quad . \tag{3.7}$$

(The relation of the momenta with our previous convention is given by $p^+ = -\dfrac{\sqrt{k+2}}{2}p$, $p^- = \dfrac{\sqrt{k+2}}{2}p - \dfrac{2m}{\sqrt{k+2}}$, $p^\rho = i\sqrt{\dfrac{2}{k}}\left(j - \dfrac{1}{2}\right)$.) Now we assume $p^+ p^- \neq 0$. The BRST transformation of this candidate (3.6) yields a non-vanishing term due to the background charge. We must cancel it by mixing the longitudinal modes to recover the BRST invariance. After some simple calculation we find *two* independent solutions

$$\left(\alpha_{-1}^\rho + i\frac{Q}{p^\pm}\alpha_{-1}^\pm\right)|p^+, p^-, p^\rho\rangle \otimes c_1|0\rangle_{\text{gh}}. \tag{3.8}$$

In the usual free string theory the general physical states are created by making the transverse DDF operators act on suitable Fock vacuum ("allowed states"). The above simple observation suggests that, in our case of AdS_3, we must make use of the two independent DDF operators that are *not purely transversal*. Some candidates for the suitable DDF operators are already given by Satoh [3][2]

$$B_n^\pm := \oint i\left(\partial\rho + \frac{Q}{2}\partial\ln\partial Y^\pm\right)e^{\frac{n}{p^\pm}iY^\pm}, \tag{3.9}$$

which are BRST invariant and satisfy the commutation relation of a free boson

$$\begin{aligned}[B_m^\pm, \ B_n^\pm] &= m\delta_{m+n,0}, \\ B_m^\pm|p^+, p^-, p^\rho\rangle &= 0, \quad (\forall m \geq 1)\end{aligned} \tag{3.10}$$

---

[2]In [3] the other candidates for the DDF operators are also proposed. However, they include the ghost number current explicitly in their expressions and are not BRST invariant. Y.S. should express his great thanks to Dr. Satoh for the discussion about this point.

Moreover, $B_n^{\pm}$ act on the Fock space defined by $|p^+, p^-, p^\rho\rangle$ as the operators $\sim \alpha_n^\rho + \cdots$, as is expected, and it is easy to check that $B_{-1}^{\pm}$ actually give rise to the first excited states discussed above (3.8). So the reader might suppose that we can naively use B_n^+, B_n^- to construct the complete set of physical states. But this is not the case. B_n^+ and B_m^- are *not mutually local* and thus we can never use them at the same time. The best we can do is to use only one of them, B_m^+, which is compatible with the light-like compactification (2.14). We rewrite it as

$$\mathcal{A}_n^{(p)} = \oint i\left(\partial\rho + \frac{Q}{2}\partial\ln\partial Y^+\right) e^{-\frac{n}{p}\frac{2}{\sqrt{k+2}}iY^+} , \quad [\mathcal{A}_m^{(p)}, \mathcal{A}_n^{(p)}] = m\delta_{m+n,0} . \tag{3.11}$$

which are defined as local operators on $\mathcal{F}_{*,*,p}$ ($p \neq 0$).

Now, the question we must solve is as follows: What is the missing DDF operator that can compensate (3.11)? As we already suggested, the answer is the space-time Virasoro operators. Let $\mathcal{L}_n$ be the space-time Virasoro operators defined in (2.25). $\mathcal{L}_{\frac{n}{p}}$ are well-defined as local operators on the Fock space $\mathcal{F}_{j,m,p}$ and behave as $\sim \alpha_n^\rho + \cdots$. Therefore they are the candidates of the missing DDF operators. Alternatively we shall define

$$\mathcal{L}_n^{(p)} := p\mathcal{L}_{\frac{n}{p}} - \frac{k+2}{4}(p^2 - 1)\delta_{n,0} , \tag{3.12}$$

which are shown to generate a Virasoro algebra with the central charge $c = 6(k+2)$ (irrespective of the value p). Furthermore, $\mathcal{L}_n^{(p)}$ are mutually local with $\mathcal{A}_m^{(p)}$ and satisfy the commutation relation

$$[\mathcal{L}_n^{(p)}, \mathcal{A}_m^{(p)}] = -m\mathcal{A}_{m+n}^{(p)} + i\alpha n^2\delta_{n+m,0}, \quad \alpha \equiv \sqrt{\frac{k}{2}}\left(1 - \frac{1}{k}\right). \tag{3.13}$$

It is also convenient to introduce improved Virasoro operators

$$\tilde{\mathcal{L}}_n^{(p)} := \mathcal{L}_n^{(p)} - \frac{1}{2}\sum_m : \mathcal{A}_{-(n+m)}^{(p)}\mathcal{A}_m^{(p)} : -i\alpha n\mathcal{A}_n^{(p)} - \frac{1}{2}\alpha^2\delta_{n,0} , \tag{3.14}$$

which are defined so that they commute with $\{\mathcal{A}_m^{(p)}\}$ and generate the Virasoro algebra with $c = 23 - \frac{6}{k} \equiv c_{\mathcal{N}}$. This value of the central charge is quite expected. One can show that the DDF operators $\tilde{\mathcal{L}}_n^{(p)}$ correspond to the energy-momentum tensor of $\mathcal{N}$ sector in the light-cone gauge.

In this way, we have found that the physical Hilbert space should be spanned by the states having the forms

$$\tilde{\mathcal{L}}_{-n_1}^{(p)}\tilde{\mathcal{L}}_{-n_2}^{(p)}\cdots\mathcal{A}_{-m_1}^{(p)}\mathcal{A}_{-m_2}^{(p)}|j, m, p\rangle \otimes \cdots , \quad (n_i \geq 1, m_i \geq 1) . \tag{3.15}$$

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In order to complete our discussion we must confirm the linear independence of the states of the type (3.15). Happily, this is very easy to prove in our case. The Virasoro algebra $\{\tilde{\mathcal{L}}_n^{(p)}\}$ has the central charge greater than 1 for sufficiently large k, and thus the Kac determinant does not vanish, as is well-known in the representation theory of Virasoro algebra.

We can now present the complete list of physical states. This spectrum is specified by the momenta of the Fock space $\mathcal{F}_{j,m,p} \otimes \bar{\mathcal{F}}_{j,\bar{m},p}$ previously defined. The light-like compactification (2.14) implies that $p \in \mathbf{Z}$, and also $\tilde{\mathcal{L}}_0^{(p)} - \overline{\tilde{\mathcal{L}}_0^{(p)}} \equiv p(\mathcal{L}_0 - \bar{\mathcal{L}}_0) \in p\mathbf{Z}$ (the "level matching condition"). Since $\mathcal{A}_0^{(p)\dagger} = \mathcal{A}_0^{(p)}$ holds and we have

$$\mathcal{A}_0^{(p)}|j, m, p\rangle = i\sqrt{\frac{2}{k}}\left(j - \frac{1}{2}\right)|j, m, p\rangle, \tag{3.16}$$

the value of j allowed by the unitarity is $j = \frac{1}{2} + is$ ($s \in \mathbf{R}$), at least when $p \neq 0$. It corresponds to the principal continuous series of unitary representation of $SL(2; \mathbf{R})$. Also in the case of $p = 0$ we can show that only the principal series is permitted from the unitarity, as we will observe below.

To avoid unnecessary complexity we shall only take the primary states in the $\mathcal{N}$ sector in constructing the physical states, and assume the conformal weights $h_{\mathcal{N}}$ of these primary states are non-negative. It is not difficult to construct more general physical states including the descendant states in the $\mathcal{N}$ sector, if we are concretely given the unitary CFT model describing this sector.

We discuss the $p = 0$ and $p \neq 0$ cases separately.

1. $p = 0$

   In this case, the DDF operators of the types $\mathcal{A}_n^{(p)}$, $\mathcal{L}_n^{(p)}$ are ill-defined. But we must require the space-time Virasoro operators $\{\mathcal{L}_n\}$ (with central charge 0) should define an unitary representation, since they are well-defined as local operators even in this sector. $\mathcal{L}_n$ simply maps a Fock vacuum to another Fock vacuum and so the representation space is given by $\oplus_{r\in\mathbf{Z}} \mathbf{C}|j, m + r, 0\rangle$ with arbitrary fixed values of $m \in \mathbf{R}$, j. We can obtain

   $$\langle 1 - j, -m - r, -p|\mathcal{L}_{-n}\mathcal{L}_n|j, m + r, p\rangle$$
   $$= \left\{\left(m + r + \frac{n}{2}\right)^2 - n^2\left(j - \frac{1}{2}\right)^2\right\}\langle 1 - j, -m - r - n, 0|j, m + r + n, 0\rangle. \tag{3.17}$$

   Since the conformal weight (in the sense of world-sheet CFT) must be real at least, j should take the values $j \in \mathbf{R}$ or $j = \frac{1}{2} + is$ ($s \in \mathbf{R}$). If $j - \frac{1}{2} \in i\mathbf{R}$ (principal series), the

coefficient of R.H.S in (3.17) is always positive and we have an unitary representation of $\{\mathcal{L}_n\}$. On the other hand, if $j \in \mathbf{R}$, we can always find $r \in \mathbf{Z}$ for which this coefficient becomes negative as long as we choose n to be sufficiently large. This means that the cases of $j \in \mathbf{R}$ cannot be unitary representations of $\{\mathcal{L}_n\}$, and hence we must rule out these sectors.

The general physical states with $p = 0$ are written as

$$
\begin{aligned}
&|\frac{1}{2} + is, m, 0\rangle \otimes |h_{\mathcal{N}}\rangle \otimes c_1|0\rangle_{\text{gh}} \\
&\otimes \quad |\frac{1}{2} + is, \bar{m}, 0\rangle \otimes |h_{\mathcal{N}}\rangle \otimes \bar{c}_1|0\rangle_{\text{gh}} \\
&m, \bar{m} \in \mathbf{R} \ , \ m - \bar{m} \in \mathbf{Z} \ ,
\end{aligned}
\tag{3.18}
$$

where $|h_{\mathcal{N}}\rangle$ is the primary state with conformal weight $h_{\mathcal{N}}$ in the $\mathcal{N}$ sector and $|0\rangle_{\text{gh}}$ is the vacuum of the ghost system. The on-shell condition is given by

$$
\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + h_{\mathcal{N}} = 1 \quad .
\tag{3.19}
$$

If $k > 1/4$, we can always solve the on-shell condition (3.19) for $h_{\mathcal{N}} = 0$. These physical states are tachyons whose mass-squared are lower than the Breitenlohner-Freedman bound [30]. Such an instability in bosonic string theory is not surprising, and we later observe that the GSO projection successfully eliminates these tachyonic states in the superstring case.

There is one comment: If we took account of the unitarity of the representation *only of* $SL(2; \mathbf{R})$ (that is, $\{\mathcal{L}_n\}$, $n = 0, \pm 1$), many representations would survive in the sector $j \in \mathbf{R}$: the discrete series $\mathcal{D}_j^{\pm}$ and the exceptional series $\mathcal{E}_{j,\alpha}$, as is well-known and many readers might expect. It is crucial in the above argument to take the *full* Virasoro algebra $\{\mathcal{L}_n\}$ in place of the $SL(2; \mathbf{R})$ subalgebra.

2. $p \neq 0$

As we already discussed, in this sector j must be equal to $j = \frac{1}{2} + is$ $(s \in \mathbf{R})$, and the physical Hilbert space is generated by the actions of the DDF operators $\{\mathcal{A}_{-n}^{(p)}\}$, $\{\tilde{\mathcal{L}}_{-n}^{(p)}\}$ $(n \in \mathbf{Z}_{\geq 1})$ on the on-shell Fock vacua. We must discuss the positivity of the norm of such physical states. Obviously $\mathcal{A}_{-n}^{(p)}$ create only positive norm states, and do not lead to any constraint. However, the Virasoro generators $\{\tilde{\mathcal{L}}_n^{(p)}\}$ give rise to a non-trivial constraint for unitarity. Since this Virasoro algebra has the central charge

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$c > 1$, the condition for the unitarity means that the $\tilde{\mathcal{L}}_0^{(p)}$-eigenvalue of the Fock vacuum $|\frac{1}{2} + is, m, p\rangle$ is non-negative. (We again assume k is sufficiently large.) It is easy to show that this unitarity condition is equivalent to a simple inequality $h_{\mathcal{N}} \geq 0$ thanks to the on-shell condition

$$\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + mp - \frac{k+2}{4}p^2 + h_{\mathcal{N}} = 1 \ . \tag{3.20}$$

This equivalence is not surprising, since $\tilde{\mathcal{L}}_0^{(p)}$ corresponds to the stress tensor for $\mathcal{N}$ sector in the light-cone gauge. In this way we conclude that the no-ghost theorem for this sector is trivially satisfied as long as the internal CFT $\mathcal{N}$ is unitary. This result is consistent with those of [2, 3], although we here take a different convention of free field representation: Our convention diagonalizes the time-like current j^3 (corresponding to the energy operator). On the other hand, those given in [2, 3, 10] diagonalize one of the space-like currents. We remark that the light-cone momentum p plays a role similar to that of the extra zero-mode momentum emphasized in [2, 3, 10].

One can find that the $\mathcal{L}_0$-eigenvalue (not $\tilde{\mathcal{L}}_0^{(p)}$) of the on-shell Fock vacuum, which corresponds to the space-time energy, is bounded below

$$\mathcal{L}_0 \geq \frac{h_{\mathcal{N}}}{p} + \frac{(k-1)^2}{4kp} + \frac{k+2}{4}\left(p - \frac{1}{p}\right) \sim \frac{h_{\mathcal{N}} - 1}{p} + \frac{k+2}{4}p, \tag{3.21}$$

(for a sufficiently large value k). This means that this sector corresponds to the long string states in the sense of [11] and belongs to the continuous spectrum above the threshold energy $\sim \frac{k}{4}p$ discussed in [14, 15].

In summary, the general physical states are written as

$$\begin{aligned}
&\tilde{\mathcal{L}}_{-n_1}^{(p)} \tilde{\mathcal{L}}_{-n_2}^{(p)} \cdots \mathcal{A}_{-m_1}^{(p)} \mathcal{A}_{-m_2}^{(p)} \cdots |\frac{1}{2} + is, m, p\rangle \otimes |h_{\mathcal{N}}\rangle \otimes c_1|0\rangle_{\mathrm{gh}} \\
&\otimes \ \overline{\tilde{\mathcal{L}}^{(p)}}_{-\bar{n}_1} \overline{\tilde{\mathcal{L}}^{(p)}}_{-\bar{n}_2} \cdots \overline{\mathcal{A}^{(p)}}_{-\bar{m}_1} \overline{\mathcal{A}^{(p)}}_{-\bar{m}_2} \cdots |\frac{1}{2} + is, \bar{m}, p\rangle \otimes |\bar{h}_{\mathcal{N}}\rangle \otimes \bar{c}_1|0\rangle_{\mathrm{gh}} \\
&\quad n_1, n_2, \cdots \geq 1, \quad m_1, m_2, \cdots \geq 1 \\
&\quad \bar{n}_1, \bar{n}_2, \cdots \geq 1, \quad \bar{m}_1, \bar{m}_2, \cdots \geq 1 \ ,
\end{aligned} \tag{3.22}$$

where the on-shell conditions are

$$\begin{aligned}
\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + mp - \frac{k+2}{4}p^2 + h_{\mathcal{N}} &= 1 \\
\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + \bar{m}p - \frac{k+2}{4}p^2 + \bar{h}_{\mathcal{N}} &= 1 \ ,
\end{aligned} \tag{3.23}$$

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and the "level matching condition" is given as

$$\sum_i n_i + \sum_j m_j - mp = \sum_i \bar{n}_i + \sum_j \bar{m}_j - \bar{m}p \pmod{p} \ . \tag{3.24}$$

The superstring case $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ is similarly analyzed. The unitarity of the physical Hilbert space is derived from the unitarity in the $N = 2$ SCFT describing $\mathcal{N}/U(1)$ sector. We here only discuss how tachyonic states in the $p = 0$ sector are eliminated by the GSO projection.

The tachyon vertex operators have slightly different expressions as compared with the bosonic case

$$V_{j,m,p} = e^{\left(\frac{\sqrt{k}}{2}p - \frac{2m}{\sqrt{k}}\right)iX^+ + \frac{\sqrt{k}}{2}piX^- - \sqrt{\frac{2}{k}}j\rho} \ . \tag{3.25}$$

Together with the vertex for the S^1 direction $e^{-i\sqrt{\frac{2}{k}}qX^2}$ we construct the Fock vacuum $|j, m, p, q\rangle$ such that

$$J_0^3|j, m, p, q\rangle = (m - \frac{k}{2}p)|j, m, p, q\rangle, \tag{3.26}$$

$$J_{\mp p}^{\pm}|j, m, p, q\rangle = (m \pm j)|j, m \pm 1, p, q\rangle, \tag{3.27}$$

$$J_{\mp p+n}^{\pm}|j, m, p, q\rangle = 0, \quad (\forall n \geq 1), \tag{3.28}$$

$$L_0|j, m, p, q\rangle = \left(-\frac{1}{k}j(j-1) + mp - \frac{k}{4}p^2 + \frac{q^2}{k}\right)|j, m, p, q\rangle, \tag{3.29}$$

$$J_{R0}'|j, m, p, q\rangle = \left(\frac{2q}{k} + p\right)|j, m, p, q\rangle. \tag{3.30}$$

In the $p = 0$ sector the argument similar to the bosonic case leads to the following physical states;

$$|\frac{1}{2} + is, m, 0, q\rangle \otimes |h_\mathcal{N}, q_\mathcal{N}\rangle \otimes ce^{-\phi}|0\rangle_{\text{gh}}$$

$$\otimes \quad |\frac{1}{2} + is, \bar{m}, 0, \bar{q}\rangle \otimes |\bar{h}_\mathcal{N}, \bar{q}_\mathcal{N}\rangle \otimes \bar{c}e^{-\bar{\phi}}|0\rangle_{\text{gh}} \tag{3.31}$$

$$m, \bar{m} \in \mathbf{R} \ , \ m - \bar{m} \in \mathbf{Z} \ ,$$

with the on-shell conditions

$$\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + \frac{q^2}{k} + h_\mathcal{N} = \frac{1}{2}$$

$$\frac{1}{k}\left(s^2 + \frac{1}{4}\right) + \frac{\bar{q}^2}{k} + \bar{h}_\mathcal{N} = \frac{1}{2} \ . \tag{3.32}$$

Naively we can solve the on-shell conditions as in the bosonic case and they are tachyonic states except for $s = 0$. However, we can show that the GSO projection eliminates such tachyonic states, as is expected. From the on-shell conditions (3.32) and the assumption $h_{\mathcal{N}} \geq \frac{1}{2}|q_{\mathcal{N}}|$, which is derived from the unitarity in the $\mathcal{N}/U(1)$-sector, we obtain the inequality

$$\frac{1}{k}(\frac{1}{4} + s^2) + \frac{q^2}{k} + \frac{|q_{\mathcal{N}}|}{2} \leq \frac{1}{2}. \tag{3.33}$$

We should define the GSO condition with respect to the deformed $U(1)_R$ current J'_R and it reads as $\frac{2q}{k} + q_{\mathcal{N}} = 2l + 1$ $(l \in \mathbf{Z})$. First we assume $q_{\mathcal{N}} \geq 0$. If $l \geq 0$, substituting $q_{\mathcal{N}} = 2l + 1 - \frac{2q}{k}$ into the above inequality (3.33), we obtain

$$s^2 + \left(q - \frac{1}{2}\right)^2 + 2lk \leq 0, \tag{3.34}$$

which leads to $n = 0$, $q = \frac{1}{2}$, $s = 0$. In the case of $l < 0$, $q \leq \frac{k}{2}(2l + 1)$ must hold. We thus obtain

$$\frac{1}{2} \geq \frac{1}{k}\left(\frac{1}{4} + s^2\right) + \frac{q^2}{k} + \frac{|q_{\mathcal{N}}|}{2} \geq \frac{s^2}{k} + \frac{1}{2}, \tag{3.35}$$

which leads to $s = 0$, again. Therefore the tachyonic states whose mass-squared are lower than the BF bound are successfully eliminated by the GSO projection. We can repeat the same analysis when $q_{\mathcal{N}} < 0$.

3.2 Physical Spectrum under the Existence of Liouville Potential

In the previous argument only the principal series was allowed. In the physical sense it was a quite natural result, because we regarded the system as a *free system* and thus all the momenta should be real.

Now, let us try to turn on the Liouville potential term (or the screening charge term in the world-sheet action (2.1)). In this case we can expect some physical states with an imaginary momentum along the ρ-direction describing the bound states ("bound string states" in the terminology of [11]).

Unfortunately, a rigorous treatment of the quantum Liouville theory as an interacting theory is quite non-trivial. Instead we shall here take the operator contents as free fields and treat the Liouville potential as a small perturbation.

Recalling the σ-model action (2.1), this perturbation term may be identified with the operator $\sim S\bar{S}$, where

$$S = \oint \beta e^{-\sqrt{\frac{2}{k}}\varphi} \equiv -\sqrt{\frac{k+2}{2}} \oint i\partial Y^1 e^{-\sqrt{\frac{2}{k}}\rho} \tag{3.36}$$

is no other than the familiar screening charge which commutes with all modes of $\widehat{SL}(2;\mathbf{R})$ currents. As for the spectrum generating operators, we may as well expect that at least $\mathcal{L}_{\pm 1}(\sim \mathcal{L}_{\pm p}^{(p)})$, $\mathcal{L}_0$ remain the good DDF operators, since they commute with the screening charge (3.36).

On the other hand, because such an interaction breaks the translational invariance along the ρ-direction, the ρ-momenta $\sim i\left(j - \frac{1}{2}\right)$ loses its meaning. However the second Casimir $\sim j(j-1)$ remains well-defined as a conserved quantity characterizing the physical states even under the interacting theory. This is nothing but the standard argument of "screening out" of the extra zero-mode momentum in the free field representation of CFT_2 [31, 8, 18]. We may expect the bound string states possessing the imaginary ρ-momenta as long as their second Casimirs take real values. In this way we can no longer regard $\mathcal{A}_0^{(p)}$ as a good DDF operator. Moreover, we must also rule out the non-zero modes $\mathcal{A}_n^{(p)}$ ($n \neq 0$), because we have the following commutation relations $\mathcal{A}_0^{(p)} \sim [(\mathcal{L}_{-1})^n, \mathcal{A}_n^{(p)}] + \text{const.}$ for $n > 0$, and $\mathcal{A}_0^{(p)} \sim [(\mathcal{L}_1)^{-n}, \mathcal{A}_n^{(p)}] + \text{const.}$ for $n < 0$.

It is a subtle problem whether the other modes of Virasoro operators $\mathcal{L}_n^{(p)}$ ($n \neq 0, \pm p$) remain the members of the spectrum generating algebra, since they also do not commute with the screening charge (3.36). However, it may be plausible to admit these operators from the point of view of the AdS_3/CFT_2 correspondence or the arguments of Brown-Henneaux [12]. The fact that only the $SL(2;\mathbf{R})$ generators $\mathcal{L}_0$, $\mathcal{L}_{\pm 1}$ commute with the screening charge and the other modes do not is supposed to reflect the following fact: In the argument of [12] the true isometry generates only the $SL(2;\mathbf{R})$ and the other modes merely correspond to the asymptotic isometries, which can be regarded as symmetries only near the boundary. We shall now propose that *the DDF operators suitable for the interacting theory including (3.36) are $\{\mathcal{L}_n^{(p)}\}$ rather than those for the free system $\{\tilde{\mathcal{L}}_n^{(p)}, \mathcal{A}_n^{(p)}\}$.* This claim is consistent with the analyses based on the light-cone gauge for the long string configuration [21, 15]. Hence our assertion is likely to be consistent at least with the spectrum of the long string located near the boundary. (One should keep it in mind that our assumption of small Liouville perturbation is valid only for such a configuration of world-sheet.)

Based on this assumption we now present the complete physical spectrum as the interacting theory. For the principal series $j = \dfrac{1}{2} + is$, the results are similar to those of free fields. The case of $p = 0$ is the same as before, and when $p \neq 0$, only we have to do is to replace the DDF operators $\{\mathcal{A}_n^{(p)}, \tilde{\mathcal{L}}_m^{(p)}\}$ in the expression of (3.22) by $\{\mathcal{L}_n^{(p)}\}$. They likewise belong to a continuous spectrum above the threshold energy $\sim \dfrac{kp}{4}$.

A crucial difference is the existence of the physical states with $j \in \mathbf{R}$ as we already suggested. To discuss the unitarity of this sector we again assume $p > 0$, and the cases of $p < 0$ can be analyzed in the same way. The unitarity condition means that the $\mathcal{L}_0^{(p)}$-eigenvalue of the Fock vacuum should be non-negative. Thanks to the on-shell condition

$$-\frac{1}{k}j(j-1) + mp - \frac{k+2}{4}p^2 + h_{\mathcal{N}} = 1 \ , \tag{3.37}$$

we can immediately obtain the following inequality for the unitarity

$$\frac{1}{k}\left(j - \frac{1}{2}\right)^2 \leq h_{\mathcal{N}} + \frac{(k-1)^2}{4k} \ . \tag{3.38}$$

We must also restrict the range of j as $j > 1/2$ because of the normalizability of wave function (see [32]). Especially, in the case of $h_{\mathcal{N}} = 0$ we obtain the unitarity condition

$$\frac{1}{2} < j \leq \frac{k}{2}. \tag{3.39}$$

These physical states do not propagate along the radial direction ρ, and are supposed to correspond to the bound string states in the argument of [11]. In fact, we can evaluate the space-time energy for this sector

$$\frac{k+2}{4}\left(p - \frac{1}{p}\right) \lesssim \mathcal{L}_0 \lesssim \frac{h_{\mathcal{N}} - 1}{p} + \frac{k+2}{4}p, \tag{3.40}$$

which is consistent with the result given in [11].

The physical states are summarized as follows;

$$\mathcal{L}_{-n_1}^{(p)} \mathcal{L}_{-n_2}^{(p)} \cdots |j, m, p\rangle \otimes |h_{\mathcal{N}}\rangle \otimes c_1 |0\rangle_{\text{gh}}$$
$$\otimes \ \overline{\mathcal{L}_{-\bar{n}_1}^{(p)} \mathcal{L}_{-\bar{n}_2}^{(p)} \cdots |j, \bar{m}, p\rangle \otimes |\bar{h}_{\mathcal{N}}\rangle \otimes \bar{c}_1 |0\rangle_{\text{gh}}}$$
$$n_1, n_2, \cdots \geq 1, \quad \bar{n}_1, \bar{n}_2, \cdots \geq 1, \tag{3.41}$$

where the on-shell conditions are

$$-\frac{1}{k}j(j-1) + mp - \frac{k+2}{4}p^2 + h_{\mathcal{N}} = 1$$
$$-\frac{1}{k}j(j-1) + \bar{m}p - \frac{k+2}{4}p^2 + \bar{h}_{\mathcal{N}} = 1 \ . \tag{3.42}$$

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and the "level matching condition" is given as

$$\sum_i n_i - mp = \sum_i \bar{n}_i - \bar{m}p \quad (\text{mod } p). \tag{3.43}$$

We here remark that the positive (negative) energy $(\mathcal{L}_0 \geq 0)$ physical states should have $p > 0$ $(p < 0)$. This fact will be important in the discussions in the next section about the interpretation of the long string theory.

To close this section we compare the above spectrum with the result of [11]. For this purpose we must clarify which representations of $\widehat{SL}(2;\mathbf{R})$ we should choose.

Let us assume $p < 0$. Going back to the Wakimoto free fields φ, β, γ we introduce the "Wakimoto module" $\mathcal{W}_{j,m,p}$ which is defined as the Fock space generated by α^φ_{-n-1}, β_{p-n}, γ_{-p-n} $(n \geq 0)$ out of the vacuum $|j, m, p\rangle$ for $m \neq j$, and by α^φ_{-n-1}, β_{p-n-1}, γ_{-p-n} $(n \geq 0)$ for $m = j$ (corresponding to the flowed discrete series $\hat{\mathcal{D}}_j^{+\,(p)}$ in [11]). $\mathcal{W}_{j,m,p}$ is obviously a subspace of $\mathcal{F}_{j,m,p}$ (and they are not isomorphic). Moreover, at least under the restriction $\frac{1}{2} < j \leq \frac{k}{2}$ (3.39), we can show that $\mathcal{W}_{j,m,p}$ can be identified with some (reducible, in general) $\widehat{SL}(2;\mathbf{R})$-module, since we have no singular vectors in the corresponding Verma module (except for the Fock vacua themselves). It is easy to see that

$$\prod_i \mathcal{L}^{(p)}_{-n_i} |j, m, p\rangle \in \bigoplus_{r \in \mathbf{Z}} \mathcal{W}_{j,m+\frac{r}{p},p}. \tag{3.44}$$

Therefore we can successfully realize the actions of DDF operators $\mathcal{L}_n^{(p)}$ within the (reducible) representations corresponding to $\bigoplus_{r \in \mathbf{Z}} \mathcal{W}_{j,m+\frac{r}{p},p}$.

For $p > 0$, the essentially same argument works by introducing the "inverse Wakimoto representation";

$$\begin{cases} j^3 = \tilde{\beta}\tilde{\gamma} - \sqrt{\dfrac{k}{2}}\partial\varphi \\ j^+ = \tilde{\beta} \\ j^- = \tilde{\beta}\tilde{\gamma}^2 - \sqrt{2k}\tilde{\gamma}\tilde{\varphi} - (k+2)\partial\tilde{\gamma}, \end{cases} \tag{3.45}$$

or more explicitly,

$$\begin{cases} \tilde{\varphi} = \rho - \sqrt{\dfrac{2k}{(k+2)}}Y^+ \\ \tilde{\beta} = \left(-\sqrt{\dfrac{k+2}{2}}i\partial Y^1 + \sqrt{\dfrac{k}{2}}\partial\rho\right) e^{-\frac{2}{\sqrt{k+2}}iY^+} \\ \tilde{\gamma} = e^{\frac{2}{\sqrt{k+2}}iY^+}. \end{cases} \tag{3.46}$$

Consequently our choice of the representations of $\widehat{SL}(2;\mathbf{R})$ is much larger than that of [11] for the cases of $j \in \mathbf{R}$, although the enlargement of the Hilbert space by the spectral flow

in [11] is incorporated into our setup as the discrete light-cone momentum p. By construction our physical Hilbert space also contains no ghosts. In the analysis in [11] m must take discrete values related with a fixed j (belonging to the discrete series of $\widehat{SL}(2;\mathbf{R})$ transformed by the spectral flow); $j - m \in \mathbf{Z}$. On the other hand, in our analysis m *is arbitrary and independent of j* as long as it satisfies the on-shell condition. This is natural from our starting point: the σ-model (2.1) rather than the abstract representation theory of affine Lie algebra, and thus j and m (and of course p, too) should correspond to independent momenta along the different directions.

Furthermore, $\hat{\mathcal{D}}_j^{+\,(p)}$ and $\hat{\mathcal{D}}_{\frac{k+2}{2}-j}^{-\,(p-1)}$ are identified in [11], since they are equivalent as an irreducible representation of $\widehat{SL}(2;\mathbf{R})$. Nevertheless they should be distinguished from our viewpoint, because they possess the different light-cone momenta p. Especially, the standard discrete series $\hat{\mathcal{D}}_j^+$ (lowest weight representations), $\hat{\mathcal{D}}_j^-$ (highest weight representations) are realized in the sectors $p = -1$, $p = 1$ respectively, since the sectors $p = 0$, $j \in \mathbf{R}$ are excluded in our analysis.

We also mention that the above unitarity condition (3.39) is analogous to the result given in [9, 11], but it is a slightly stronger condition. The unitarity bound proposed in [9] reads $\frac{1}{2} < j < \frac{k+2}{2}$ and the one given in [11] reads $\frac{1}{2} < j < \frac{k+1}{2}$ in our convention[3]. This disagreement originates from the different choices of the representations mentioned above. In fact, if one choose to restrict the value of m as $j - m \in \mathbf{Z}$ when solving the on-shell condition, one can show the no-ghost theorem under the assumption $\frac{1}{2} < j < \frac{k+2}{2}$ same as [9] rather than (3.39)[4]. It is not yet clear whether our choice of the momenta m, independent of j, is completely valid even in the rigid treatment as an interacting theory, or nothing but an artifact originating from the free field approximation. However, we again emphasize that our setup of physical Hilbert space admits the whole actions of DDF operators $\{\mathcal{L}_n^{(p)}\}$ (and necessarily also with the space-time Virasoro algebra $\{\mathcal{L}_n\}$). This fact is found to be consistent with the several results about the long string sectors given by the light-cone gauge approach [21, 15, 17], as we will comment in the next section. In fact, one can readily find

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[3]The unitarity bound for [11] is stronger than that of [9] due to the identification $\hat{\mathcal{D}}_j^{+\,(p)} = \hat{\mathcal{D}}_{\frac{k+2}{2}-j}^{-\,(p-1)}$ mentioned above. Moreover, the same range of j was proposed in a different context [29] by requiring good behaviors of the two point functions of the non-normalizable primary operators. Such two point functions nicely behave, too, under our constraints (3.39), since it is more stringent than that of [29].

[4]To be precise, under this restriction $j - m \in \mathbf{Z}$ we must take $\mathcal{L}_n$ as the DDF operators rather than $\mathcal{L}_n^{(p)} = p\mathcal{L}_{n/p} + \cdots$, (recall (3.44)) and hence the unitarity here means that $\mathcal{L}_n$-descendants should not include any negative norm states.

that our choice of m so as to be *independent of j* is crucial in order to ensure the equivalence with the spectrum in the light-cone gauge after solving the on-shell condition. In this sense we believe that our physical spectrum is valid at least for the long string configuration near the boundary, which is well described by the light-cone gauge approach. In order to justify this spectrum beyond the near boundary region we will have to carry out a further analysis with the Liouville interaction term treated more precisely.

The extension of the above arguments to superstring examples is not so difficult and we do not present it here. We instead focus on the spectrum of on-shell chiral primaries of superstring on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ in the next section.

4 Chiral Primaries and Spectral Flow

In this section we further study the spectrum in the superstring cases. We especially investigate an important class of observables: chiral primaries. In other words, we shall concentrate on the topological sector of superstring vacua on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ [33]. They are significant from the perspective of AdS_3/CFT_2-duality. Although we have not yet achieved the complete understanding of this duality, the study of their spectrum will certainly clarify some aspects of it.

Through this section we only deal with the left moving parts of objects, and it is easy to complete our discussions by taking also the right movers.

4.1 Background with Space-time $N = 4$ SUSY

We first discuss the most familiar superstring vacua with space-time $N = 4$ SUSY;

$$AdS_3 \times S^3 \times T^4 \cong AdS_3 \times S^1 \times SU(2)/U(1) \times T^4, \tag{4.1}$$

where $SU(2)/U(1)$ means that the Kazama-Suzuki model [25] for this coset with $c = 3 - \dfrac{6}{N}$ ($N - 2$ is equal to the level of (bosonic) $SU(2)$-WZW model describing the S^3 sector), which is identified with the $N = 2$ minimal model of A_{N-1} type and we denote it by M_N from

now on. The criticality condition gives $k = N$, and this background is regarded as the near horizon limit of $NS5/NS1$ system, as is well-known[4, 5].

Let $|\Phi_l\rangle$ $(l = 0, 1, \ldots, N - 2)$ be the chiral primary states in the M_N sector with $h_{\mathcal{N}} = \frac{q_{\mathcal{N}}}{2} = \frac{l}{2N}$. We must look for the chiral primary states in the total system by tensoring $|\Phi_l\rangle$ with suitable vertex operators in the $AdS_3 \times S^1$ sector. In the notation of previous section, namely,

$$|j, m, p, q\rangle \equiv \lim_{z \to 0} e^{\left(\frac{\sqrt{k}}{2}p - \frac{2m}{\sqrt{k}}\right)iX^+ + \frac{\sqrt{k}}{2}piX^- - \sqrt{\frac{2}{k}}j\rho - \sqrt{\frac{2}{k}}qiX^2} |0\rangle, \tag{4.2}$$

the possible candidates for the desired vertices are written as follows;

$$|j, j, p, j - \frac{k}{2}p\rangle, \quad \Psi_{-1/2}^+ |j, -(j - 1), p, -(j - 1) - \frac{k}{2}p\rangle. \tag{4.3}$$

They are primary states with respect to $T(z)$, $G(z)$ and also satisfy

$$G_{-1/2}^+ |j, j, p, j - \frac{k}{2}p\rangle = G_{-1/2}^+ \Psi_{-1/2}^+ |j, -(j - 1), p, -(j - 1) - \frac{k}{2}p\rangle = 0. \tag{4.4}$$

First we consider $|j, j, p, j - \frac{k}{2}p\rangle \otimes |\Phi_l\rangle \otimes c_1 e^{-\phi}|0\rangle_{\text{gh}}$. The on-shell condition leads to

$$j = \frac{N - l}{2}. \tag{4.5}$$

(The GSO condition is automatically satisfied, since we have the relation $h = \frac{Q}{2}$.) Similarly, for the second candidate $\Psi_{-1/2}^+ |j, -(j - 1), p, -(j - 1) - \frac{k}{2}p\rangle \otimes |\Phi_l\rangle \otimes c_1 e^{-\phi}|0\rangle_{\text{gh}}$ we can solve the on-shell condition and obtain

$$j = 1 + \frac{l}{2}. \tag{4.6}$$

From now on we denote the first type of chiral primary (4.5) as $|l, p; 1\rangle$ and the second type (4.6) as $|l, p; 2\rangle$. Namely, we set

$$|l, p; 1\rangle := |\frac{N - l}{2}, \frac{N - l}{2}, p, -\frac{N(p - 1) + l}{2}\rangle \otimes |\Phi_l\rangle \otimes ce^{-\phi}|0\rangle_{\text{gh}} \tag{4.7}$$

$$|l, p; 2\rangle := \Psi_{-1/2}^+ |\frac{l}{2} + 1, -\frac{l}{2}, p, -\frac{Np + l}{2}\rangle \otimes |\Phi_l\rangle \otimes ce^{-\phi}|0\rangle_{\text{gh}} \ . \tag{4.8}$$

They are both normalizable and satisfy the unitarity condition (3.38).

Remarkably one can find that (4.7), (4.8) are also chiral primaries with respect to the space-time superconformal algebra $\{\mathcal{L}_n, \mathcal{G}_r^\pm, \mathcal{J}_n\}$, as suggested in [33]. They satisfy

$$\mathcal{L}_0 |l, p; 1\rangle = \frac{1}{2}\mathcal{J}_0 |l, p; 1\rangle = \frac{1}{2}\{l + N(p - 1)\}|l, p; 1\rangle \tag{4.9}$$

$$\mathcal{L}_0 |l, p; 2\rangle = \frac{1}{2}\mathcal{J}_0 |l, p; 2\rangle = \frac{1}{2}(l + Np)|l, p; 2\rangle \ . \tag{4.10}$$

Since the light-cone momentum p can now take an arbitrary integer, we have infinite number of on-shell chiral primaries. All of them have the same $U(1)_R$ charge in the sense of world-sheet because of the the on-shell condition. But they have the different $U(1)_R$ charges with respect to the space-time conformal algebra.

Let us study the action of the spectral flow on these states. A natural extension of the spectral flow (2.23) to the superstring case is given by

$$U_p X^0(z) U_p^{-1} = X^0(z) - p\sqrt{\frac{k}{2}} i \ln z$$

$$U_p X^2(z) U_p^{-1} = X^2(z) + p\sqrt{\frac{k}{2}} i \ln z, \tag{4.11}$$

and the other fields should remain unchanged by the spectral flow. The total $\widehat{SL}(2;\mathbf{R})$-currents are transformed by them as follows;

$$U_p J_n^3 U_p^{-1} = J_n^3 + \frac{kp}{2}\delta_{n,0}$$

$$U_p J_n^\pm U_p^{-1} = J_{n\mp p}^\pm . \tag{4.12}$$

We can show that

$$U_r |l, p\, ; i\rangle = |l, p+r\, ; i\rangle. \tag{4.13}$$

Namely, the spectral flow maps an on-shell chiral primary to another on-shell chiral primary. This is a general feature. In fact, it is a straightforward calculation to check that

$$U_p G^+(z) U_p^{-1} = G^+(z) \tag{4.14}$$

$$U_p Q_{BRST} U_p^{-1} = Q_{BRST} - \lim_{z\to 0} \left[pc\sqrt{\frac{k}{2}} i\partial(X^0 + X^2) + p\eta e^\phi \sqrt{\frac{k}{2}}(\Psi^0 + \Psi^2) \right]. \tag{4.15}$$

Because the correction term in (4.15) vanishes when acting on an arbitrary on-shell chiral primaries, the spectral flow U_p preserves the on-shell condition in the space of chiral primaries. One should remark that U_p does *not* transform all the physical states among themselves. Indeed the physical states which are not the chiral primaries are transformed into off-shell states by U_p.

Turning our attention to the space-time conformal algebra, we obtain

$$\{\mathcal{G}_{\frac{1}{2}}^-, \mathcal{G}_{-\frac{1}{2}}^+\} = \mathcal{L}_0 - \frac{1}{2}\mathcal{J}_0$$

$$= -\oint \sqrt{\frac{k}{2}} i\partial(X^0 + X^2) , \tag{4.16}$$

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and this identity is unchanged by the spectral flow. This means that the spectral flows are closed in the space of the *space-time* chiral primaries, which is consistent with the above observation.

Next we present some remarks from the view points of AdS_3/CFT_2-duality and the long string theory given in [15]. Although our understanding of them by string theory is not yet complete, we believe the following remarks are useful to clarify some important aspects of them.

Tracing back to the procedure of our field redefinitions, one can find that $|l, p = 1 ; 1\rangle$ can be identified with the space-time chiral primary states given in [34] (see also [35])[5]. It has the following structure

$$|l, 1 ; 1\rangle = \lim_{z \to 0} \mathcal{O}_l(z)|0, 1 ; 1\rangle, \tag{4.17}$$

where

$$\mathcal{O}_l(z) := V_{-\frac{l}{2}, -\frac{l}{2}, 0, -\frac{l}{2}} \Phi_l(z) \equiv e^{\frac{l}{\sqrt{2N}}(iX^0 + iX^1 + iX^2 + \rho)} \Phi_l(z) \tag{4.18}$$

naturally corresponds to the chiral operator in the light-cone gauge formalism of the long string theory given in [15]. $|0, 1 ; 1\rangle \equiv e^{-\sqrt{\frac{N}{2}}(iX^1 + \rho)} c e^{-\phi}|0\rangle$ has the maximal j-value $j = N/2 \equiv k/2$ and is the same as the "space-time vacuum" (or "long string vacuum") presented in [34]. It satisfies

$$\mathcal{L}_n|0, 1 ; 1\rangle = 0, \quad (\forall n \geq -1), \tag{4.19}$$

$$\mathcal{G}_r^{\pm}|0, 1 ; 1\rangle = 0, \quad (\forall r \geq -1/2), \tag{4.20}$$

$$\mathcal{J}_n|0, 1 ; 1\rangle = 0, \quad (\forall n \geq -1), \tag{4.21}$$

$$-\frac{\sqrt{k}}{2} \oint i\partial X^+|0, 1 ; 1\rangle = |0, 1 ; 1\rangle, \tag{4.22}$$

where the last line simply means that $p \equiv \oint \gamma^{-1}\partial\gamma = 1$, and these identities hold up to BRST-exact terms.

As discussed in [15] (see also [32]), the chiral operator $\mathcal{O}_l$ corresponds to a non-normalizable state. Its wave function is divergent near the boundary, where the Coulomb branch CFT is weakly coupled. On the other hand, thanks to the existence of $|0, 1 ; 1\rangle$, the chiral primary state $|l, 1 ; 1\rangle$ itself becomes normalizable state vanishing exponentially at large ρ, as expected

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[5] They correspond to $|\omega_{l/2}^0\rangle$ in the notation of [34], which contain the trivial cohomology in the T^4 sector. We can also consider more general *space-time* chiral primary states that contain higher cohomologies of T^4, as given in [34]. But they are not chiral primaries in the sense of world-sheet.

from the observation about the Higgs branch tube in [15]. In this sense the interpretation of $|0, 1 ; 1\rangle$ as the long string vacuum may be natural.

It may be also useful to define explicitly the *space-time* chiral primary operator $\hat{\mathcal{O}}_l(x) \equiv \sum_n \dfrac{\hat{\mathcal{O}}_{l,n}}{x^{n+\frac{l}{2}}}$ by introducing the vertex operators

$$\hat{\mathcal{O}}_{l,n} := \oint V_{-\frac{l}{2},n,0,-\frac{l}{2}} \Phi_l (\Psi^0 + \Psi^1) e^{-\phi} , \tag{4.23}$$

where n runs over all (half-)integers if $\dfrac{l}{2}$ is an (half-)integer. $\hat{\mathcal{O}}_l(x)$ actually behaves as a chiral primary operator with respect to the space-time SCA. For example, we obtain

$$[\mathcal{L}_m, \hat{\mathcal{O}}_{l,n}] = \left\{ (\frac{l}{2} - 1)m - n \right\} \hat{\mathcal{O}}_{l,m+n}, \tag{4.24}$$

which means that $\hat{\mathcal{O}}_l(x)$ is a primary operator with conformal weight $h = \dfrac{l}{2}$. We can further show that

$$\hat{\mathcal{O}}_{l,n}|0, 1 ; 1\rangle = 0, \quad (\forall n > -\frac{l}{2}) \tag{4.25}$$

and also obtain the "operator-state correspondence"

$$\hat{\mathcal{O}}_{l,-\frac{l}{2}}|0, 1 ; 1\rangle = |l, 1 ; 1\rangle \tag{4.26}$$

(up to the picture changing and an overall constant).

For $p > 1$ we can consider more general chiral primaries with the higher space-time $U(1)_R$-charges

$$|l, p ; 1\rangle = \mathrm{U}_{p-1}|l, 1 ; 1\rangle = \mathcal{O}_l(0)|0, p ; 1\rangle = \hat{\mathcal{O}}_{l,-\frac{l}{2}}|0, p ; 1\rangle . \tag{4.27}$$

As we observed above (4.9), the spectrum of $\mathcal{J}_0$ charge is $l + N(p - 1)$, $l = 0, 1, \ldots, N - 2$, $p \geq 1$.

We have not yet known the suitable interpretation of the "graviton-like" chiral primary states $|l, p ; 2\rangle$ in the context of AdS_3/CFT_2 correspondence. We only point out that they do not seem to have the forms such as (4.26), and so it might be plausible to suppose that they do not have any counterparts in the boundary theory, as long as our identification of $|0, 1 ; 1\rangle$ with the space-time vacuum is justified. In any case we will need a further analysis to give a more definite statement about this problem.

The following aspect may be worthwhile to point out. Here the chiral primaries with the higher space-time $U(1)_R$-charges appear in the sector with higher light-cone momenta p (or

by taking account of the degrees of freedom of spectral flows). On the other hand, in the analysis of [17], they correspond to the $\mathbf{Z}_p$-twisted sector of the symmetric orbifold theory, which describes the sector of long string with the "length" p as in the Matrix string theory [37]. It suggests a remarkable correspondence between the spectral flow in the covariant gauge formalism and the twisted sector of the symmetric orbifold in the light-cone gauge formalism [21, 15].

To address the precise correspondence between them we should work on the *second quantized* framework. It is quite reasonable from the viewpoints of AdS_3/CFT_2 correspondence, since the boundary CFT should also contain multi-particle excitations. We shall now focus on the physical states with positive energies, which should have the light-cone momenta $p \geq 0$ as we found in section 3. The physical Hilbert space of the first quantized string states, which was studied in our previous analyses, is then decomposed to $\mathcal{H} = \bigoplus_{p \geq 0} \mathcal{H}_p$, where $\mathcal{H}_p$ denotes the sector with the light-cone momentum $p \geq 0$. The Hilbert space in the (free) second quantized theory can be roughly written as

$$\hat{\mathcal{H}} = \bigoplus_{n=0}^{\infty} (\mathcal{H})^{\otimes n}. \tag{4.28}$$

(To be precise, we must make some (anti-)symmetrization to assure the correct statistics in this and the expressions given below.) The second quantized space $\hat{\mathcal{H}}$ has a natural decomposition with respect to the total light-cone momentum

$$\hat{\mathcal{H}} = \bigoplus_{p \geq 0} \hat{\mathcal{H}}_p, \tag{4.29}$$

Obviously $\hat{\mathcal{H}}_p$ is decomposed to the subspaces of the forms

$$\mathcal{H}_{p_1} \otimes \mathcal{H}_{p_2} \otimes \cdots \otimes \mathcal{H}_{p_n} \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \cdots, \quad \left(p_1 \geq p_2 \geq \ldots \geq p_n \geq 1, \ \sum_{i=1}^{n} p_i = p\right). \tag{4.30}$$

The $p = 0$ Hilbert space $\mathcal{H}_0$ only contains tachyons, which are eliminated by the GSO projection, as was already shown[6]. Therefore we can neglect the $\mathcal{H}_0$ factors, and can explicitly write down

$$\hat{\mathcal{H}}_p = \bigoplus_{n=1}^{p} \bigoplus_{\sum_{i=1}^{n} p_i = p} (\otimes_{i=1}^{n} \mathcal{H}_{p_i}) . \tag{4.31}$$

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[6]The fact that the physical Hilbert space of "short string sector" $\mathcal{H}_0$ is vacant is not a contradiction. It rather means that only the non-normalizable physical *operators* can appear and the operator-state correspondence fails in the short string sector as suggested in [6].

Now let us consider the system of Q_5 NS5 and Q_1 NS1. The NS5 charge $Q_5(\equiv N)$ appears in the world-sheet action of AdS_3-string theory, but Q_1 does not. It only appears in the string coupling, which is stable under the near horizon limit, and hence we cannot find this effect in the first quantized theory. However, in the second quantized theory, it is quite natural to identify the NS1 charge Q_1 with the total light-cone momentum p in the expression of $\hat{\mathcal{H}}_p$. Hence we propose that the physical Hilbert space of this NS5-NS1 system should be defined as $\hat{\mathcal{H}}_{Q_1}$. Notice that it has the structure characterized by the various partitions $\{p_i(\geq 1)\,; \sum_i p_i = Q_1\}$ which is consistent with the expected correspondence to the symmetric orbifold theory. Clearly this system can be decomposed to the subsystems of various long strings with the "lengths" (or "windings") p_i ($1 \leq p_i \leq Q_1 \sum_i p_i = Q_1$), as observed in [15, 7].

It is interesting to present the spectrum of the "single-particle chiral primaries" in this framework. Let $p \leq Q_1$ be a positive integer. We obtain the required states as

$$|l,p\,;1\rangle \otimes \underbrace{|0,1\,;1\rangle \otimes \cdots \otimes |0,1\,;1\rangle}_{Q_1 - p\text{-times}} \ , \tag{4.32}$$

which satisfies $\mathcal{L}_0 = \dfrac{1}{2}\mathcal{J}_0 = \dfrac{l + Q_5(p-1)}{2}$ as expected. Since l, p run over the ranges $0 \leq l \leq Q_5 - 2$, $0 \leq p \leq Q_1$, this spectrum is completely in agreement with the result of [17], in which the (multiple) long string CFT was analyzed using the symmetric orbifold theory. Quite remarkably, this has the upper bound $\sim \dfrac{Q_1 Q_5}{2}$ which is expected from the AdS_3/CFT_2-duality [36], as already commented in [17].

Notice that there are the missing states corresponding to the $\mathcal{J}_0$-charge $Q_5 p - 1$ ($1 \leq p \leq Q_1$). They are (formally) written as

$$|Q_5 - 1, p\,;1\rangle \otimes |0,1\,;1\rangle \otimes \cdots = \mathcal{O}_{Q_5-1}(0)|0,p\,;1\rangle \otimes |0,1\,;1\rangle \otimes \cdots \ , \tag{4.33}$$

and "$\mathcal{O}_{Q_5-1}(z)$" is no other than the missing chiral operator discussed in [15], which should correspond to the cohomology with a delta function support at the small instanton singularity. In this sense these missing states (4.33) are supposed to be the natural generalizations to the cases of $1 < p \leq Q_1$ of the one discussed in [15], in which only the $p = 1$ sector was treated.

The above observation implies that the first quantized Hilbert space $\mathcal{H}_p$ ($p \leq Q_1$) precisely corresponds to the $\mathbf{Z}_p$-twisted sector in the S_{Q_1}-orbifold theory as we already suggested. The relation

$$\mathcal{L}_n = \frac{1}{p}\mathcal{L}_{pn}^{(p)} + \frac{Q_5}{4}\left(p - \frac{1}{p}\right)\delta_{n,0} \tag{4.34}$$

indeed confirms this identification. On the Hilbert space $\mathcal{H}_p$, the space-time Virasoro algebra $\mathcal{L}_n$ has the central charge $c = 6pQ_5$, and the DDF operators $\mathcal{L}_n^{(p)}$ generate the Virasoro algebra with $c = 6Q_5$. This relation (4.34) is the same as the well-known formula to define the conformal algebra describing the $\mathbf{Z}_p$-twisted sector of the symmetric orbifold. It is easy to define the tensor product representation of space-time conformal algebra with $c = 6Q_1Q_5$ on the second quantized Hilbert space $\hat{\mathcal{H}}_{Q_1} \equiv \bigoplus_{\sum_i p_i = Q_1} (\otimes \mathcal{H}_{p_i})$ including the conformal invariant vacuum

$$\underbrace{|0,1\,;1\rangle \otimes |0,1\,;1\rangle \otimes \cdots \otimes |0,1\,;1\rangle}_{Q_1 \text{times}} \in \mathcal{H}_1^{\otimes Q_1} \quad . \tag{4.35}$$

Such a correspondence, which was essentially suggested in [21, 17], is quite expected from our standpoint as the discrete light-cone theory fitted to the spirit of Matrix string [23, 37]. Recall that our setup of physical Hilbert space in section 3 allows the action of $\mathcal{L}_n^{(p)} = p\mathcal{L}_{n/p} + \cdots$, and moreover we must impose the "level matching condition" $\mathcal{L}_0^{(p)} - \overline{\mathcal{L}_0^{(p)}} \in p\mathbf{Z}$ onto the Hilbert space $\mathcal{H}_p$. These facts are crucial to establish the above correspondence to the symmetric orbifold.

One should keep in mind the following fact: one can also construct the representation with $c = 6Q_1Q_5$ on the first quantized Hilbert space $\mathcal{H}_{Q_1}$ that is the subspace of $\hat{\mathcal{H}}_{Q_1}$ describing the *single* long string with the maximal length Q_1. *However, $\mathcal{H}_{Q_1}$ cannot include the conformal invariant vacuum.* Recall that $\mathcal{L}_0|0,p\,;1\rangle \neq 0$, unless $p = 1$. More precisely speaking, we can show that, in our setup of the first quantized Hilbert space the BRST-invariant state with the properties (4.19), (4.20), (4.21) and non-zero p is possible only if $p = 1$, and the solution is unique (up to BRST exact terms and an overall normalization), $|0,1\,;1\rangle$, as suggested in [34, 17]. This fact leads us to the only one possibility of the conformal invariant vacuum (4.35). The large Hagedorn density suited for $c = 6Q_1Q_5$, which may reproduce the correct entropy formula of black-hole, should be attached to $\hat{\mathcal{H}}_{Q_1}$, *not* to $\mathcal{H}_{Q_1}$, since $\mathcal{H}_{Q_1}$ does not include the vacuum state such that $\mathcal{L}_0 = 0$ (see the discussions given in [32, 38]).

4.2   Background with Space-time $N = 2$ SUSY

In principle it is not difficult to generalize the above analysis on chiral primaries to more general superstring vacua with space-time $N = 2$ SUSY [24].

We first give a rather generic argument. Consider superstring theory on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$, where $\mathcal{N}/U(1)$ is an arbitrary $N = 2$ SCFT of center $9 - 6/k$. As in the $N = 4$ case, we can construct two series of chiral primary states from chiral primaries V_j of conformal weight $j/2k$ in the $\mathcal{N}/U(1)$-sector:

$$|j, p\,; 1\rangle \ := \ |\frac{k-j}{2}, \frac{k-j}{2}, p, -\frac{k(p-1)+j}{2}\rangle \otimes |V_j\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}} \tag{4.36}$$

$$|j, p\,; 2\rangle \ := \ \Psi^+_{-1/2}|\frac{j+2}{2}, -\frac{j}{2}, p, -\frac{kp+j}{2}\rangle \otimes |V_j\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}} \tag{4.37}$$

They have the light-cone momentum p and the following conformal weight:

$$\mathcal{L}_0|j, p\,; 1\rangle = \frac{1}{2}\mathcal{J}_0|j, p\,; 1\rangle \ = \ \frac{j+k(p-1)}{2}|j, p\,; 1\rangle$$

$$\mathcal{L}_0|j, p\,; 2\rangle = \frac{1}{2}\mathcal{J}_0|j, p\,; 2\rangle \ = \ \frac{j+kp}{2}|j, p\,; 2\rangle \tag{4.38}$$

Note that, if we take as $\mathcal{N}/U(1)$ an arbitrary $N=2$ SCFT of center $9-6/k$, the conformal weight $h = j/2k$ of V_j runs within the range $0 \leq h \leq 3 - 2/k$. However, it is only if $0 \leq h \leq 1/2 - 1/k$ that the chiral primary states are in the spectrum allowed from unitarity and normalizability.

Let us consider a specific example. Take as $\mathcal{N}/U(1)$ the $N = 2$ minimal model which we denote by M_N as before. It was proposed in [40] that the superstring theory on this background is marginally equivalent to the non-critical superstring theory [26] which is holographically dual to the decoupled theory based on the A_{N-1}-singular CY_4. In this case the criticality condition leads to $k = \dfrac{N}{N+1}$.

Let $|\Phi_l\rangle$ $(l = 0, 1, \ldots, N - 2)$ be again the chiral primary states of weight $l/2N$ in the M_N sector. We obtain the following chiral primaries;

$$|l, p\,; 1\rangle \ := \ \left|\frac{N-l}{2(N+1)}, \frac{N-l}{2(N+1)}, p, -\frac{N(p-1)+l}{2(N+1)}\right\rangle \otimes |\Phi_l\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}} \tag{4.39}$$

$$|l, p\,; 2\rangle \ := \ \Psi^+_{-1/2}\left|\frac{l}{2(N+1)} + 1, -\frac{l}{2(N+1)}, p, -\frac{Np+l}{2(N+1)}\right\rangle \otimes |\Phi_l\rangle \otimes ce^{-\phi}|0\rangle_{\mathrm{gh}}. \tag{4.40}$$

In this way we have again infinite number of on-shell chiral primaries possessing the following space-time $U(1)_R$ charges;

$$\mathcal{J}_0|l, p\,; 1\rangle \ = \ \left(\frac{l}{N+1} + \frac{N}{N+1}(p-1)\right)|l, p\,; 1\rangle \tag{4.41}$$

$$\mathcal{J}_0|l, p\,; 2\rangle \ = \ \left(\frac{l}{N+1} + \frac{N}{N+1}p\right)|l, p\,; 2\rangle. \tag{4.42}$$

Unfortunately, $|l, p\,; 1\rangle$ is non-normalizable and $|l, p\,; 2\rangle$ does not satisfy the unitarity constraints (3.38). Hence we cannot consider the chiral primary *states* within the physical Hilbert space. We must only treat these chiral primaries as *operators* and cannot expect the operator-state correspondence. Nevertheless, they may be regarded as an important class of operators in the context of AdS_3/CFT_2-duality, or more general holographic dualities [39, 40]. In particular the non-normalizable chiral primaries $|l, p\,; 1\rangle$ ("tachyon-like operators") may be important, because they possess the momentum structures which can be regarded as natural generalizations of those of the scaling operators in the space-time conformal theory proposed in [40]. Since the light-cone momentum p runs over an infinite range, we can obtain the infinite tower of space-time chiral operators for each of the chiral operators in the "matter sector" Φ_l. This aspect may be interesting, since they look like analogues of "gravitational descendants" in the theory of two dimensional gravity. We must make further studies to gain more precise insights about these objects. In addition, the roles of the graviton-like operators $|l, p\,; 2\rangle$ are again unclear. More detailed argument for them will be surely important, although it is beyond the scope of this paper.

5   Conclusions and Discussions

In this paper we have studied the spectrum of the physical states in string theory on AdS_3 based on a free field realization. We have found that the system is quite simply described as a linear dilaton theory with a light-like compactification, which we called as "discrete light-cone Liouville theory". Our key idea is to utilize the DDF operators according to the traditional approach to string theory on flat backgrounds. We have *two* independent sets of DDF operators; $\mathcal{A}_n^{(p)}$, $\tilde{\mathcal{L}}_n^{(p)}$. This situation is similar to the non-critical string, although we started with the *critical* string on $AdS_3 \times \mathcal{N}$ background. In fact, we can easily find that a suitable linear combination of $\mathcal{A}_n^{(p)}$ and $\tilde{\mathcal{L}}_n^{(p)}$ gives the "longitudinal DDF operator" utilized in [41].

Regarding the system as a free theory with no Liouville interaction term (screening charge term), the physical spectrum contains only the principal series $j = \dfrac{1}{2} + is$ as asserted in [2, 3, 10], and the physical states are generated by the DDF operators $\{\mathcal{A}_n^{(p)}, \tilde{\mathcal{L}}_n^{(p)}\}$ mentioned above.

However, once we turn on the Liouville potential, the story becomes rather non-trivial. In this interacting theory the translational invariance along the Liouville direction is broken. The physical Hilbert space is expected to be spanned only by the $\widehat{SL}(2;\mathbf{R})$ currents, rather than the whole oscillators of the string coordinates ρ, Y^+, Y^-, because the interaction term commutes only with the $\widehat{SL}(2;\mathbf{R})$ currents. In this situation the spectrum generating algebra becomes $\{\mathcal{L}_n^{(p)}\}$ in place of $\{\mathcal{A}_n^{(p)}, \tilde{\mathcal{L}}_n^{(p)}\}$, and the physical states possessing the imaginary ρ-momenta ($j \in \mathbf{R}$) are also allowed. Physically they correspond to the bound string states of [11] that are trapped inside the AdS_3-space.

It may be worthwhile to mention that only the physical Hilbert space as the interacting Liouville theory may be consistent with the microscopic evaluation of the black hole entropy. In the free system the DDF operators should be $\{\mathcal{A}_n^{(p)}, \tilde{\mathcal{L}}_m^{(p)}\}$ and $\tilde{\mathcal{L}}_m^{(p)}$ (which are identified with the Virasoro operators in $\mathcal{N}$-sector under the light-cone gauge) have a small central charge $c = 23 - \dfrac{6}{k}$. (An important discussion related to such a counting of physical states was given in [38].) On the other hand, after turning on the Liouville potential we claimed that the full Virasoro generators $\{\mathcal{L}_n^{(p)}\}$ are well-defined (and it is also crucial that $\{\mathcal{A}_n^{(p)}\}$ should be discarded). Taking further account of the second quantized Hilbert space they seem to generate sufficiently many physical states with the Hagedorn density that can reproduce the correct entropy. We would like to study this problem in more detail elsewhere.

In the study of superstring examples, we have presented the complete set of on-shell chiral primaries. There exist infinite number of such operators and the spectral flows naturally act on them. Moreover, to describe the well-known $Q_5(\equiv k)$ NS5 - Q_1 NS1 system we made use of the second quantized framework. The Hilbert space of the multiple long string system was given as the form $\hat{\mathcal{H}}_{Q_1} = \bigoplus\limits_{\sum_i p_i = Q_1} (\otimes_i \mathcal{H}_{p_i})$, where $\mathcal{H}_p$ denotes the first quantized physical Hilbert space of the sector with the light-cone momentum p (> 0). This space reproduce the spectrum of chiral primaries same as that given by the symmetric orbifold theory [17], and among other things, we have successfully obtained the upper bound $\sim Q_1 Q_5/2$ consistent with the prediction of AdS_3/CFT_2 correspondence [36].

It may be also worth pointing out that our reformulation of superstring on $AdS_3 \times S^1 \times \mathcal{N}/U(1)$ has the same field contents as those of the non-critical string that is holographically dual to a singular Calabi-Yau compactification (especially, the cases of CY_4) proposed in [40]. The only difference between these models is the existence/absence of the light-like compactification. In [40] it was discussed that these two backgrounds can be interpolated by some

marginal deformation. It may be an interesting problem to clarify the rigid correspondence between them. In particular, our analyses on general chiral operators in section 4 will be readily generalized to the cases of such non-critical string theories.

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